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GEOMETRY OF SCHUBERT CELLS AND COHOMOLOGY OF KAC-MOODY LIE-ALGEBRAS

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0. Introduction

Let $\mathbf{q} = \mathbf{q}(A)$ be the Kac-Moody Lie algebra associated to a symmetrizable generalized Cartan matrix $A = (a_{ij})_{1 \leqslant i,j \leqslant l}$ and let $S \subset \{1,\cdots,l\}$ be a subset of finite type. Denote by $\mathbf{p}_S$ the corresponding "parabolic"; $\mathbf{r}$ "the maximal" reductive subalgebra of $\mathbf{p}_S$ and $\mathbf{u}^-$ the orthocomplement of $\mathbf{p}_S$ (so that $\mathbf{u}^- \oplus \mathbf{p}_S = \mathbf{q}$). Let $L(\lambda_0)$ be the quasi-simple $\mathbf{q}$-module, with highest weight λ_0. Further, let $(\Lambda(\mathbf{u}^-, L(\lambda_0)^t), \partial)$ be the standard chain complex associated to the Lie algebra $\mathbf{u}^-$ with coefficients in the right module $L(\lambda_0)^t$, and let $(C(\mathbf{q}, \mathbf{r}), d)$ denote the standard cochain complex associated to the Lie algebra pair $(\mathbf{q}, \mathbf{r})$ (with trivial coefficient $\mathbf{C}$). There is associated (in general infinite dimensional) a group G (resp. a "parabolic" subgroup P_S) with $\mathbf{q}^1$ (resp. $\mathbf{p}_S \cap \mathbf{q}^1$). The flag variety G/P_S admits a Bruhat cell decomposition with cells $\{V_w\}$ parametrized by $w \in W_S \setminus W \cong W_S^1$ (W is the Weyl group for $\mathbf{q}$).

In this paper we explicitly compute the action of the Laplacian $\Delta = \partial\partial^* + \partial^*\partial$ on $\Lambda(\mathbf{u}^-, L(\lambda_0)^t)$. Further, we use this to prove the "disjointness" of the operators d and ∂ (defined in §3), acting on $C(\mathbf{q}, \mathbf{r})$. This gives rise to a "Hodge type" decomposition, with respect to the pair d, ∂ (d, ∂ are not adjoints of each other), of the space $C(\mathbf{q}, \mathbf{r})$. In particular, every d cohomology class in $C(\mathbf{q}, \mathbf{r})$ has a unique d, ∂ closed representative. The "Hodge type" decomposition also gives, by a slight refinement of the arguments, that $H_d^*(\mathbf{q}, \mathbf{r})$ is bigraded;

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$H_d^{p,q}(\mathbf{q}, \mathbf{r}) = 0$ unless $p = q$ and $H_d^{p,p}(\mathbf{q}, \mathbf{r})$ is a vector space with a "canonical" **C**-basis $\{s^w\}_{w \in W_S^1 \text{ with length } w = p}$. Finally (and this was one of our main points of interest) we prove that, properly defined, $\int_{V_{w'}} s^w = 0$ unless $w = w'$ and $\int_{V_w} s^w > 0$. So the d, ∂ harmonic forms $\{s^w\}_{w \in W_S^1}$, properly normalized, are dual to the Bruhat cells of the flag variety G/P_S. This result, in particular, applies to the based (at e) loop space $\Omega_e(K_0)$ of a finite-dimensional compact connected simply connected Lie group K_0. Garland and Raghunathan had conjectured the existence of such a dual basis $\{s^w\}$, in this case. The topology of $\Omega_e(K_0)$ has been studied extensively by Bott via Morse theory.

When $\mathbf{q}$ is a finite-dimensional semisimple Lie algebra, all these results are due to Kostant and are contents of his papers [16] and [17]. In the infinite dimensional situation, Garland has computed Δ for a special case. Most of the other results are new (as far as is known to the author).

The author's indebtedness to various ideas in Kostant's two important papers [16] and [17] would be clear to any informed reader.

Now, we describe the contents of this paper in more detail.

In §1 we recall the (well) known (though scattered) facts from Kac-Moody Lie algebras, which we would, often, be using. We also fix some notations to be used throughout the paper.

Let $\mathbf{q} = \mathbf{q}(A)$ be the Kac-Moody Lie algebra associated to a symmetrizable generalized Cartan matrix A and let $S \subset \{1, \cdots, l\}$ be a subset of finite type. In §2 we briefly define $\mathbf{r}$, $\mathbf{u}^-$, $L(\lambda_0)$ and the chain complex $(\Lambda(\mathbf{u}^-, L(\lambda_0)^t), \partial)$ in §1. The main theorem of this section (Theorem (2.1)) describes the action of the Laplacian $\Delta = \partial\partial^* + \partial^*\partial$ on $\Lambda(\mathbf{u}^-, L(\lambda_0)^t)$. More precisely, the theorem states: *Let W^β be an irreducible* $\mathbf{r}$*-submodule of* $\Lambda(\mathbf{u}^-, L(\lambda_0)^t)$ *with highest weight* β. *Then* $\Delta|_{W^\beta}$ *is scalar multiplication by* $\frac{1}{2}[\sigma(\lambda_0 + \rho, \lambda_0 + \rho) - \sigma(\beta + \rho, \beta + \rho)]$. We would also like to isolate Remark (2.2) and Proposition (2.10). Remark (2.2) gives an (equivalent) "invariant" reformulation of the main theorem, e.g., when $L(\lambda_0)$ is the one-dimensional trivial module, it says that the Laplacian 2Δ is "essentially" negative of the Casimir operator. Proposition (2.10) describes the action of a "Casimir like" operator on $\mathbf{u}^-$. When $\mathbf{q}$ is finite dimensional, this proposition is well known and is trivial to prove. Of course, as an important corollary (Corollary (2.3)(a)) of Theorem (2.1), we can deduce one of the main theorems of Garland-Lepowsky in [6], which describes the homology $H_*(\mathbf{u}^-, L(\lambda_0)^t)$.

In the infinite-dimensional situation; Garland [5] has computed the action of the Laplacian in the special case when $\mathbf{q}$ is an affine Lie algebra; $\mathbf{p}$ is the standard maximal F-parabolic and $L(\lambda_0)$ is the one-dimensional trivial module. His computation (unpublished) is fairly long and complicated.

The proof of our Theorem (2.1) differs, in many ways, from Kostant's original proof. Trying to generalize Kostant's proof as it is, one encounters many difficulties, most notably; one encounters various infinite (often meaningless) sums (e.g. $\Sigma_{\varphi \in \Delta} \operatorname{ad} y_\varphi \operatorname{ad} x_\varphi$ acting on $\mathbf{u}^-$!).

To conclude, the amount of cancellations (occurring in the proof of the theorem) has some parallels with the proof of the Index Theorem given by V. K. Patodi, which involved an enormous amount of cancellations. It is not known if a proof of Theorem (2.1) can be given similar in spirit to the proof of the Index Theorem evolved after works of Gilkey and Patodi (see [1]). The author is indebted to Professor H. Garland for some of these thoughts.

In §3 the operators d', d'', ∂' and ∂'', acting on $C(\mathbf{q}, \mathbf{r})$, are defined. d' (resp. d'') corresponds to the holomorphic (resp. antiholomorphic) differential and $d = d' + d''$. $\partial = \partial' + \partial''$ is obtained by transporting the differential of the chain-complex $\Lambda(\mathbf{u} \oplus \mathbf{u}^-)$, corresponding to the Lie algebra $\mathbf{u} \oplus \mathbf{u}^-$ (defining $[\mathbf{u}, \mathbf{u}^-] = 0$), via the map e induced by the Killing form. The main theorems of this section are Theorems (3.11) and (3.15). Theorem (3.11) states that the pair (d, ∂) (resp. (d', ∂'); (d'', ∂'')), acting on $C(\mathbf{q}, \mathbf{r})$, is disjoint (in the sense of Kostant). The content of Theorem (3.15) is: 1) $H(C(\mathbf{q}, \mathbf{r}), d)$ is bigraded, 2) $H^{p,q}(C(\mathbf{q}, \mathbf{r}), d) \approx H^{p,q}(C(\mathbf{q}, \mathbf{r}), d'')$, 3) $H^{p,q}(C(\mathbf{q}, \mathbf{r}), d) = 0$ if $p \neq q$ and $H^{p,p}(C(\mathbf{q}, \mathbf{r}), d)$ is a vector space of $\dim/\mathbf{C}$ equal to the number of Weyl group elements in W_S^1 of length p. (The properties 1) and 2) are shared by all the compact Kahler manifolds!). In fact, more strongly, there is available a "Hodge type" decomposition of $C(\mathbf{q}, \mathbf{r})$, with respect to the operators (d, ∂). This is the content of our Theorem (3.13). In particular, this gives that in every d-cohomology class, there is a unique form s which is both d and ∂ closed. This fact will be used in §4.

In the general case, Lepowsky has computed the cohomology $H_d^*(\mathbf{q}, \mathbf{r})$ by a different method (see [18, Corollary 6.7]). But his results neither give information about the bigraded cohomology groups nor do they give the existence of a d, ∂ closed form in every cohomology class (a fact which will be very crucially used, as in [17], in the next section). Actually in our Remark (3.3), we have indicated a very simple proof to recover [18, Corollary 6.7].

The proofs in this section are fairly along the lines of [17], except that some of them require modifications and correct formulations. As in [17], we have used the computation of the Laplacian done in the previous section, in the special case when $L(\lambda_0)$ is the one-dimensional trivial module. We would like to specifically mention one difficulty arising in the case when $\mathbf{q}$ is infinite dimensional. This is Lemma (3.8), stating that $\operatorname{Ker} S \oplus \operatorname{Im} S = C(\mathbf{q}, \mathbf{r})$, where $S = d\partial + \partial d$. This is trivial in the case, $\mathbf{q}$ is finite-dimensional, but requires more subtle argument in the infinite dimensional case. We have introduced a "natural" topology on $C(\mathbf{q}, \mathbf{r})$, which helps to simplify the arguments.

A word is in order. We could have omitted proofs of some of the lemmas in this section (referring the reader to [17]), but we decided to present them here for clarity and completeness.

In §4, the main theorem is Theorem (4.5). To describe this, there is a Bruhat decomposition (see §4.1) for the flag variety G/P_S with cells $\{V_w\}_{w \in W_S^1}$, where $\dim_{\mathbf{R}} V_w = 2.\text{length}\, w$. Further to any $w \in W_S^1$, there is associated a d, ∂ harmonic form $s^w \in C^{2.\text{length}\, w}(\mathbf{q}, \mathbf{r})$, obtained by the disjointness of d, ∂ and using the $\mathbf{r}$-module structure of $H_*(\mathbf{u}^-, \mathbf{C})$. Now, Theorem (4.5) states that, defined appropriately, $\int_{V_{w'}} s^w = 0$ unless $w = w'$ and $\int_{V_w} s^w > 0$.

In the infinite-dimensional case, at least for the "based loop groups" (which are flag varieties associated to the affine groups) this was conjectured by Garland-Raghunathan (see the last paragraph in [7]). *In fact this was one of the main motivations behind the whole paper.*

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1. Preliminaries and notations

We recall some facts, now fairly known, about Kac-Moody Lie algebras which we will be using frequently in our paper. (See, for the details, [6], [10]–[14], [18], [19]).

1.1 Definitions. (a) A *symmetrizable generalized Cartan* matrix $A = \{a_{ij}\}_{1 \leqslant i, j \leqslant l}$ is a matrix of integers satisfying $a_{ii} = 2$ for all i, $a_{ij} \leqslant 0$ if $i \neq j$, DA is symmetric for some diagonal matrix $D = \text{dia}(q_1, \cdots, q_l)$ with $q_i > 0 \in \mathbf{Q}$.

(b) Choose a triple $(\mathbf{h}, \pi, \pi^V)$, unique up to isomorphism, where $\mathbf{h}$ is a vector space over $\mathbf{C}$ (the field of complex numbers) of dimension $l + \text{corank}\, A$, $\pi = \{\alpha_i\}_{1 \leqslant i \leqslant l} \subset \mathbf{h}^*$ and $\pi^V = \{h_i\}_{1 \leqslant i \leqslant l} \subset \mathbf{h}$ are linearly independent indexed sets satisfying $\alpha_j(h_i) = a_{ij}$. The *Kac-Moody algebra* $\mathbf{q} = \mathbf{q}(A)$ is the Lie algebra over $\mathbf{C}$, generated by $\mathbf{h}$ and the symbols e_i and f_i $(1 \leqslant i \leqslant l)$ with the defining relations $[\mathbf{h}, \mathbf{h}] = 0$; $[h, e_i] = \alpha_i(h)e_i$, $[h, f_i] = -\alpha_i(h)f_i$ for $h \in \mathbf{h}$ and all $1 \leqslant i \leqslant l$; $[e_i, f_j] = \delta_{ij}h_j$ for all $1 \leqslant i, j \leqslant l$; $(\text{ad}\, e_i)^{1-a_{ij}}(e_j) = 0 = (\text{ad}\, f_i)^{1-a_{ij}}(f_j)$ for all $1 \leqslant i \neq j \leqslant l$.

$\mathbf{h}$ is canonically embedded in $\mathbf{q}$.

1.2 Root space decomposition. There is available the root space decomposition $\mathbf{q} = \mathbf{h} \oplus \sum_{\alpha \in \Delta \subset \mathbf{h}^*} \mathbf{q}_\alpha$, where $\mathbf{q}_\alpha = \{x \in \mathbf{q}: [h, x] = \alpha(h)x$, for all $h \in \mathbf{h}\}$ and $\Delta = \{\alpha \in \mathbf{h}^* - \{0\}$ such that $\mathbf{q}_\alpha \neq 0\}$. Moreover $\Delta = \Delta_+ \cup \Delta_-$, where $\Delta_+ \subset \{\sum_{i=1}^{l} n_i\alpha_i: n_i \in \mathbf{Z}_+$ (= the nonnegative integers) for all $i\}$ and $\Delta_- = -\Delta_+$. Elements of Δ_+ (resp. Δ_-) are called positive (resp. negative) roots.

1.3 Parabolics. We fix a subset S (including $S = \varnothing$) of $\{1, \cdots, l\}$ of finite type, i.e. the submatrix $A_S = \{a_{ij}\}_{i,j \in S}$ is a classical Cartan matrix of finite type. There is a natural injection $\mathbf{q}_S = \mathbf{q}(A_S) \hookrightarrow \mathbf{q}(A)$. Define Δ_+^S (resp. Δ_-^S) = $\Delta_+ \cap \{\sum_{i \in S} \mathbf{Z}\alpha_i\}$ (resp. $\Delta_- \cap \{\sum_{i \in S} \mathbf{Z}\alpha_i\}$). Then

$$\mathbf{q}_S = \mathbf{h}_S \oplus \sum_{\alpha \in \Delta_+^S} \mathbf{q}_\alpha \oplus \sum_{\alpha \in \Delta_-^S} \mathbf{q}_\alpha,$$

where $\mathbf{h}_S$ = linear span of $\{h_i\}_{i \in S}$. Define the following Lie subalgebras.

$$\mathbf{n} = \sum_{\alpha \in \Delta_+} \mathbf{q}_\alpha; \qquad \mathbf{n}^- = \sum_{\alpha \in \Delta_-} \mathbf{q}_\alpha;$$

$$\mathbf{u} = \sum_{\alpha \in \Delta_+ \setminus \Delta_+^S} \mathbf{q}_\alpha; \quad \mathbf{u}^- = \sum_{\alpha \in \Delta_- \setminus \Delta_-^S} \mathbf{q}_\alpha;$$

$$\mathbf{r} = \mathbf{q}_S + \mathbf{h}; \qquad \mathbf{p} = \mathbf{r} \oplus \mathbf{u}.$$

Of course, $\mathbf{q} = \mathbf{h} \oplus \mathbf{n} \oplus \mathbf{n}^- = \mathbf{u}^- \oplus \mathbf{p}$ and $\mathbf{r}$ is a reductive algebra. $\mathbf{p}$ is called the F-parabolic subalgebra (F for finite-dimensionality of $\mathbf{q}_S$), defined by S (see [6, §3]). If $S = \varnothing$, the associated parabolic $\mathbf{p}$ ($= \mathbf{h} \oplus \mathbf{n}$) is the "Borel" subalgebra. If A itself is of finite type (i.e. A is a classical Cartan matrix), then the F-parabolic subalgebras are precisely the parabolic subalgebras of $\mathbf{q}$ containing the Borel subalgebra $\mathbf{h} \oplus \mathbf{n}$.

1.4 Weyl group. There is a *Weyl group* $W \subset \mathrm{Aut}(\mathbf{h}^*)$ generated by the reflections $\{r_i\}_{1 \leqslant i \leqslant l}$ $(r_i(\varphi) = \varphi - \varphi(h_i)\alpha_i)$, associated to the Lie algebra $\mathbf{q}$. $(W, \{r_i\}_{1 \leqslant i \leqslant l})$ is a Coxeter system, hence we can talk of the lengths of elements of W.

W preserves Δ. Δ^{re} is defined to be $W \circ \pi$ and $\Delta^{\mathrm{im}} = \Delta \setminus \Delta^{\mathrm{re}}$. For $\alpha \in \Delta^{\mathrm{re}}$, $\dim \mathbf{q}_\alpha = 1$ and $\Delta \cap \mathbf{Z}\alpha = \{\alpha, -\alpha\}$.

Given a subset S of finite type as in (1.3), there is defined a subset W_S^1, of the Weyl group W, by $W_S^1 = \{w \in W: \Delta_+ \cap w\Delta_- \subset \Delta_+ \setminus \Delta_+^S\}$.

1.5 Cartan involution. There is a (C-linear) unique involution ω of $\mathbf{q}$ defined by $\omega(f_i) = -e_i$ for all $1 \leqslant i \leqslant l$ and $\omega(h) = -h$ for all $h \in \mathbf{h}$. It is easy to see that ω leaves $\mathbf{q}(\mathbf{R})$ (="real points" of $\mathbf{q}$) stable.

Further, there is a unique *conjugate-linear* involution ω_0 of $\mathbf{q}$ which coincides with ω on $\mathbf{q}(\mathbf{R})$.

1.6 Killing form. There exists a nondegenerate, **q**-invariant, symmetric **C**-bilinear form $\langle\ ,\ \rangle$ on **q**, described as follows.

First define $\sigma(\alpha_i, \alpha_j) = q_i a_{ij}$ for $1 \leqslant i, j \leqslant l$. Set $h_{\alpha_i} = q_i h_i$. It is possible to extend σ to a symmetric bilinear form, again denoted by σ on $\mathbf{h}^*$, satisfying:

(1) σ is W-invariant.

(2) $\sigma(\lambda, \alpha_i) = \lambda(h_{\alpha_i})$ for all $1 \leqslant i \leqslant l$ and all $\lambda \in \mathbf{h}^*$.

We fix one such σ. Now define $\langle h_{\alpha_i}, h_{\alpha_j}\rangle = \sigma(\alpha_i, \alpha_j)$. The form $\langle\ ,\ \rangle$ on π^V can be extended to a nondegenerate, **q**-invariant, symmetric **C**-bilinear form, again denoted by $\langle\ ,\ \rangle$ on **q**, satisfying:

(1) $\langle x, y\rangle = 0$ for $x \in \mathbf{q}_\alpha, y \in \mathbf{q}_\beta$ with $\alpha, \beta \in \Delta \cup \{0\}$ and $\alpha + \beta \neq 0$.

(2) $[x, y] = \langle x, y\rangle h_\alpha$ for $x \in \mathbf{q}_\alpha$, $y \in \mathbf{q}_{-\alpha}$ and $\alpha = \sum_{i=1}^{l} n_i \alpha_i \in \Delta_+$. ($h_\alpha = \sum n_i h_{\alpha_i}$.)

We fix, once and for all, one such form. The symmetric form $\langle\ ,\ \rangle$ on **q**, described above, gives rise to a hermitian form $\{\ ,\ \}$ on **q**, defined by $\{x, y\} = -\langle x, \omega_0(y)\rangle$ for $x, y \in \mathbf{q}$. The hermitian form $\{\ ,\ \}$ is positive definite on $\mathbf{n}^-$ (and **n**). (see [22, §12] and [13, Remark IV, p. 1782]).

1.7 The Casimir operator ([12, §2.3] and [6, §4]). Let $\mathcal{O}$ denote the full category of all the (left) **q**-modules M (modules will be left, unless stated), satisfying:

(1) M is a weight module whose weight spaces are finite dimensional, i.e. $M = \sum_{\lambda \in \mathbf{h}^*} M_\lambda$ with all M_λ being finite dimensional, where $M_\lambda = \{m \in M: h \cdot m = \lambda(h)m \text{ for all } h \in \mathbf{h}\}$. λ is called a *weight* of M if $M_\lambda \neq (0)$.

(2) Let $D(M)$ be the set of all the weights of M. Then there exists a finite subset $\{\lambda_1, \cdots, \lambda_n\} \subset \mathbf{h}^*$ such that $D(M) \subset \bigcup_k D(\lambda_k)$, where $D(\lambda_k) = \{\lambda_k - \sum_{i=1}^{l} \mathbf{Z}_+ \alpha_i\}$.

Fix a $\rho \in \mathbf{h}^*$ satisfying $\rho(h_i) = 1$ for all $1 \leqslant i \leqslant l$.

There exists a natural transformation $\Gamma = \Gamma^{\mathbf{q}}$, called the *Casimir operator*, of the category $\mathcal{O}$ (i.e. given a $M \in \mathcal{O}$, there is a **q**-module map Γ_M: $M \to M$, satisfying $f \circ \Gamma_M = \Gamma_N \circ f$ for any **q**-modules M, N and a **q**-morphism f: $M \to N$).

For $\alpha \in \Delta_+$, define $\omega_\alpha = \sum t^k s^k \in U(\mathbf{q})$ (the universal enveloping algebra), where $\{s^k\}$ is any basis of $\mathbf{q}_\alpha$ and $\{t^k\}$ is the dual basis of $\mathbf{q}_{-\alpha}$ with respect to $\langle\ ,\ \rangle$. Γ is defined as $\Gamma_1 + \Gamma_2$, where Γ_1 is the operator $2\sum_{\alpha \in \Delta_+} \omega_\alpha$ (although an infinite sum, this makes sense for modules in the category $\mathcal{O}$) and Γ_2 acts on the weight space M_λ as scalar multiplication by $\sigma(\lambda + \rho, \lambda + \rho) - \sigma(\rho, \rho)$.

The action of the Casimir $\Gamma_{M^{\lambda_0}}$ on a highest weight module M^{λ_0}, with highest weight λ_0, is through scalar multiplication by $\sigma(\lambda_0 + \rho, \lambda_0 + \rho) - \sigma(\rho, \rho)$.

1.8 Quasi-simple modules. A **q**-module L is called *quasi-simple* if it is a highest weight module with highest weight vector x_0 such that there exists $n \in \mathbf{Z}_+$ with $f_i^n(x_0) = 0$ for all $1 \leqslant i \leqslant l$.

Let D be the set of all the dominant integral elements β in $\mathbf{h}^*$ (i.e. $\beta(h_i) \in \mathbf{Z}_+$ for all $1 \leqslant i \leqslant l$).

Though we would not be needing, the quasi-simple **q**-modules are indexed by D (in a bijective manner) (see [11, Corollary] and [6, Corollary 9.8]). The correspondence is given by attaching to a quasi-simple module its highest weight.

We denote by $L(\lambda)$ the quasi-simple **q**-module with highest weight $\lambda \in D$.

1.9 Algebraic group associated to a Kac-Moody Lie algebra q ([**13**], [**14**] **and** [**21**]). A $\mathbf{q}^1$ ($= [\mathbf{q},\mathbf{q}]$) module (V, π) (π: $\mathbf{q}^1 \to \operatorname{End} V$) is called integrable if $\pi(e)$ is locally nilpotent whenever $e \in \mathbf{q}_\alpha$ for $\alpha \in \Delta^{\mathrm{re}}$. Let G^* be the free product of the additive groups $\{\mathbf{q}_\alpha\}_{\alpha \in \Delta^{\mathrm{re}}}$, with canonical inclusions i_α: $\mathbf{q}_\alpha \to G^*$. For any integrable $\mathbf{q}^1$-module (V, π), define a homomorphism π^*: $G^* \to \operatorname{Aut}_{\mathbf{C}} V$ by $\pi^*(i_\alpha(e)) = \exp(\pi(e))$ for $e \in \mathbf{q}_\alpha$. Let N^* be the intersection of all $\ker(\pi^*)$. Put $G = G^*/N^*$. Let q be the canonical homomorphism $G^* \to G$. For $e \in \mathbf{q}_\alpha$ ($\alpha \in \Delta^{\mathrm{re}}$), put $\exp e = q(i_\alpha e)$, so that $U_\alpha = \exp \mathbf{q}_\alpha$ is an additive one parameter subgroup of G. Denote by U (resp. U^-) the subgroup of G generated by the U_α (resp. $U_{-\alpha}$), $\alpha \in \Delta_+^{\mathrm{re}}$.

Choose $\Lambda_i \in \mathbf{h}^*$ ($1 \leqslant i \leqslant l$) satisfying $\Lambda_i(h_j) = \delta_{ij}$ for all $1 \leqslant j \leqslant l$. There is an embedding [14, p. 162–163]

$$i \colon G \hookrightarrow \mathfrak{A} = \left(\bigoplus_{i=1}^{l} L(\Lambda_i) \right) \oplus \left(\bigoplus_{i=1}^{l} L^*(\Lambda_i) \right)$$

defined by $i(g) = g(\Sigma_{i=1}^l v_{\Lambda_i}) + g(\Sigma_{i=1}^l v_{\Lambda_i}^*)$.

Here $(L(\Lambda_i), \pi(\Lambda_i))$ is the quasi-simple module defined in §1.8; $L^*(\Lambda_i)$ is the vector space $L(\Lambda_i)$ regarded as a **q**-module under $\pi^*(\Lambda_i) = \pi(\Lambda_i) \circ \omega$; v_{Λ_i} is a highest weight vector in $L(\Lambda_i)$ and v_{Λ_i} is denoted by $v_{\Lambda_i}^*$ when regarded as an element in $L^*(\Lambda_i)$.

By "differentiating" i, we get an embedding $\bar{i}$: $\mathbf{q}^1 \to \mathfrak{A}$. More explicitly $\bar{i}(x) = x(\Sigma_{i=1}^l v_{\Lambda_i}) + x(\Sigma_{i=1}^l v_{\Lambda_i}^*)$ for $x \in \mathbf{q}^1$.

$\mathfrak{A}$ is endowed with a Hausdorff topology defined as follows. A set $V \subset \mathfrak{A}$ is open if and only if $V \cap F$ is open in F for all the finite-dimensional vector subspaces F of $\mathfrak{A}$. Now, put the subspace (through i) topology on G. G may be viewed as a, possibly infinite-dimensional, affine algebraic group in the sense of Šafarevič [20] with Lie algebra $\mathbf{q}^1$. For a proof, see [14, §4].

1.10 Notation. Throughout the paper, unless otherwise specifically stated, all the vector spaces will be over $\mathbf{C}$; linear maps will be $\mathbf{C}$-linear maps; *tensor products* and *exterior products will be over* $\mathbf{C}$. For two vector spaces V and W, $\operatorname{Hom}(V, W)$ would mean $\operatorname{Hom}_{\mathbf{C}}(V, W)$. The $\mathbf{q}$-module maps between $\mathbf{q}$-modules V and W would be denoted by $\operatorname{Hom}_{\mathbf{q}}(V, W)$. $\Lambda(V)$ denotes the exterior algebra.

For a Lie algebra pair $(\mathbf{q}, \mathbf{z})$ and a (left) $\mathbf{q}$-module X: (1) $\Lambda(\mathbf{q}, \mathbf{z}, X^t)$ will denote the standard chain complex associated to the pair, with coefficients in the right module X^t. We follow the sign convention as in [6]. (X^t is the right $\mathbf{q}$-module, whose underlying space is X and on which $\mathbf{q}$ acts by the rule $x \cdot b = -b \cdot x$ for all $b \in \mathbf{q}$ and $x \in X$); (2) $C(\mathbf{q}, \mathbf{z}, X)$ will denote the standard cochain complex associated to the pair, with coefficients in X. (see, e.g., [9, §1]).

$\Lambda(\mathbf{q}, \varnothing, X^t)$ (resp. $C(\mathbf{q}, \varnothing, X)$) will be abbreviated to $\Lambda(\mathbf{q}, X^t)$ (resp. $C(\mathbf{q}, X)$).

For a Lie algebra $\mathbf{q}$, $U(\mathbf{q})$ *denotes its universal enveloping algebra.*

2. Computation of the Laplacian for some "nilpotent" Lie algebras with coefficients in a quasi-simple module.

Let $A = \{a_{ij}\}_{1 \leq i, j \leq l}$ be a symmetrizable generalized Cartan matrix and S a subset of $\{1, \cdots, l\}$ of finite type. We have defined $\mathbf{q} = \mathbf{q}(A)$, $\mathbf{p}, \mathbf{u}, \mathbf{u}^-$ and $\mathbf{r}$ in §§1.1–1.3. Let $L(\lambda_0)$ be the quasi-simple $\mathbf{q}$-module with highest weight λ_0 (see §1.8). This admits a $\mathbf{k} = \{x \in \mathbf{q}: \omega_0(x) = x\}$ (ω_0 is defined in §1.5) invariant positive definite Hermitian form. This result is due to Garland [22, §12] in the affine case. The general case is similar and is due to Kac-Peterson. We fix one such. Then, there is a canonical Hermitian form $\{\ ,\ \}$ on $\Lambda(\mathbf{u}^-) \otimes L(\lambda_0)$.

Let ∂: $\Lambda(\mathbf{u}^-) \otimes L(\lambda_0) \to \Lambda(\mathbf{u}^-) \otimes L(\lambda_0)$ be the differential (of degree -1) of the chain complex $\Lambda(\mathbf{u}^-, L(\lambda_0)^t)$. Denote the adjoint of ∂ with respect to $\{\ ,\ \}$ by ∂^*. Existence of ∂^* will be clear when we come to the proof of the next theorem. (Notice that $\Lambda(\mathbf{u}^-) \otimes L(\lambda_0)$ is not complete!)

Since $[\mathbf{r}, \mathbf{u}^-] \subset \mathbf{u}^-$, the reductive Lie algebra $\mathbf{r}$ acts on $\mathbf{u}^-$ by adjoint action and acts on $L(\lambda_0)$ as restriction and hence $\Lambda(\mathbf{u}^-) \otimes L(\lambda_0)$ is an $\mathbf{r}$-module. It is easy to see that ∂ is an $\mathbf{r}$-module map and hence so is ∂^* (use $\mathbf{k}$ invariance of $\{\ ,\ \}$; see §1.6). Define the Laplacian, as usual, by

$$\Delta = \partial\partial^* + \partial^*\partial.$$

$\Lambda(\mathbf{u}^-) \otimes L(\lambda_0)$ decomposes as a direct sum of finite-dimensional irreducible $\mathbf{r}$-modules (see, e.g., [6, Proposition 6.3]). Now, we can state the main theorem of this section.

2.1 Theorem. *Let* $\mathbf{q} = \mathbf{q}(A)$ *be the Kac-Moody Lie algebra associated to a generalized symmetrizable Cartan matrix* $A = (a_{ij})_{1 \leq i,j \leq l}$ *and let* S *be a subset of* $\{1, \cdots, l\}$ *of finite type. Let* $L(\lambda_0)$ *be the quasi-simple* $\mathbf{q}$*-module with highest weight* λ_0*. Then, with the notations as above, the action of the Laplacian* Δ *on* $\Lambda(\mathbf{u}^-, L(\lambda_0)^t)$ *is as follows.*

Let W^β *be an irreducible* $\mathbf{r}$*-submodule of* $\Lambda(\mathbf{u}^-, L(\lambda_0)^t)$*, with highest weight* β*, then* Δ *reduces to a scalar on* W^β *and the scalar is* $\frac{1}{2}[\sigma(\lambda_0 + \rho, \lambda_0 + \rho) - \sigma(\beta + \rho, \beta + \rho)]$.

The bilinear form σ on $\mathbf{h}^*$ and ρ are defined in §§1.6 and 1.7.

2.2 Remark. Following is an equivalent reformulation of the theorem, which seems interesting.

Extend the $\mathbf{r}$-module structure on $\Lambda(\mathbf{u}^-) \otimes L(\lambda_0)$ to a $\mathbf{p}$-module structure by letting $\mathbf{u}$ act on $\Lambda(\mathbf{u}^-) \otimes L(\lambda_0)$ as the identically zero homomorphism. Consider the tensor product $U(\mathbf{q}) \otimes'_{U(\mathbf{p})} [\Lambda(\mathbf{u}^-) \otimes L(\lambda_0)]$ ($\otimes'$ is to distinguish the $U(\mathbf{p})$ action on $\Lambda(\mathbf{u}^-) \otimes L(\lambda_0)$, just given, from the $U(\mathbf{p})$ action on $\Lambda(\mathbf{u}^-) \otimes L(\lambda_0) \approx \Lambda(\mathbf{q}/\mathbf{p}) \otimes L(\lambda_0)$ used frequently in the sequel). It may be of interest, at this point, to see [6, Proposition 6.4]. Of course, $U(\mathbf{q}) \otimes'_{U(\mathbf{p})} [\Lambda(\mathbf{u}^-) \otimes L(\lambda_0)]$ breaks up as a (possibly infinite) direct sum of highest weight modules. Now, an equivalent reformulation of the theorem is

$$2\,\mathrm{Id} \otimes' \Delta = -\Gamma^{\mathbf{q}}_{(U(\mathbf{q}) \otimes'_{U(\mathbf{p})} [\Lambda(\mathbf{u}^-) \otimes L(\lambda_0)])} + \mathrm{Id} \otimes' \left(\mathrm{Id} \otimes \Gamma^{\mathbf{q}}_{L(\lambda_0)}\right),$$

as maps from

$$U(\mathbf{q}) \otimes'_{U(\mathbf{p})} [\Lambda(\mathbf{u}^-) \otimes L(\lambda_0)] \to U(\mathbf{q}) \otimes'_{U(\mathbf{p})} [\Lambda(\mathbf{u}^-) \otimes L(\lambda_0)].$$

In the case when $L(\lambda_0)$ is the trivial module $\mathbf{C}$, it says that 2Δ is "essentially" negative of the Casimir operator.

2.3 Corollaries. (a) *As the most important corollary of the above theorem, one recovers the following important theorem of Garland-Lepowsky in full generality.*

Theorem [6, Theorem 8.6]. *With the notations as in Theorem* (2.1), *the jth homology space* $H_j(\mathbf{u}^-, L(\lambda_0)^t)$ *is finite dimensional and when equipped with the standard* $\mathbf{r}$*-module action, it is naturally* $\mathbf{r}$*-module isomorphic to the direct sum*

$$\sum_{\substack{w \in W_S^1 \text{ with} \\ \text{length } w = j}} \mathbf{M}(w(\lambda_0 + \rho) - \rho)$$

of inequivalent irreducible $\mathbf{r}$*-modules. Actually, for any* $w \neq w' \in W_S^1$*, the irreducible modules* $M(w(\lambda_0 + \rho) - \rho)$ *and* $M(w'(\lambda_0 + \rho) - \rho)$ *are inequivalent.*

(See §§1.4 and 1.6 for the definitions of W_S^1, length w and ρ. Further, $M(w(\lambda_0 + \rho) - \rho)$ is the finite-dimensional irreducible $\mathbf{r}$-module with highest weight $(w(\lambda_0 + \rho) - \rho)$, which remains irreducible as a $\mathbf{q}_S$-module as well. See, for more details, [6, Proposition 3.1].)

In fact, the chain complex $\Lambda(\mathbf{u}^-, L(\lambda_0)^t)$ decomposes as the direct sum of two subcomplexes $B = \sum_{j\geqslant 0} B_j$ and $B' = \sum_{j\geqslant 0} B'_j$, such that $\partial|_B$ is identically zero; $H_*(B') = 0$ and B_j can be taken to be the sum of all the irreducible $\mathbf{r}$-submodules $M(\lambda)$ of $\Lambda^j(\mathbf{u}^-, L(\lambda_0)^t) = \Lambda^j(\mathbf{u}^-) \otimes L(\lambda_0)$ such that $\sigma(\lambda + \rho, \lambda + \rho) = \sigma(\lambda_0 + \rho, \lambda_0 + \rho)$ (with $\lambda \in \mathbf{h}^*$ satisfying $\lambda(h_i) \in \mathbf{Z}_+$ for all $i \in S$).

Of course, as is well known, $H^j(\mathbf{u}^-, L(\lambda_0)^*)$ is canonically isomorphic with the contragredient $\mathbf{r}$-module $\operatorname{Hom}(H_j(\mathbf{u}^-, L(\lambda_0)^t), \mathbf{C})$.

Proof (*of the corollary*). Define $B = \ker \Delta$ and $B' = \operatorname{Image} \Delta$. Then, of course, $\partial|_B = 0$ and $H_*(B') = 0$. The expression for Δ, stated in the theorem, gives that B_j is the sum of all the irreducible $\mathbf{r}$-submodules $M(\lambda)$ of $\Lambda^j(\mathbf{u}^-) \otimes L(\lambda_0)$ such that $\sigma(\lambda + \rho, \lambda + \rho) = \sigma(\lambda_0 + \rho, \lambda_0 + \rho)$.

Hence, $H_j(\mathbf{u}^-, L(\lambda_0)^t) \approx H_j(B) \approx B_j$. Now using [6, Propositions 8.3 and 8.4], the corollary follows.

(b) *One can specialize Theorem* (2.1) *to the case when* $S = \varnothing$ (*so that* $\mathbf{u}^- = \mathbf{n}^-$), *to get the action of the Laplacian* Δ *on* $\Lambda(\mathbf{n}^-, L(\lambda_0)^t)$.

(c) *One can specialize Theorem* (2.1) *to the case when A is a standard affine Cartan matrix*, $\mathbf{p}$ *is the standard maximal F-parabolic and* $\lambda_0 = 0$ (*so that* $L(\lambda_0) = \mathbf{C}$) *to recover Garland* [5, *Theorem* 2.5].

(d) *When we specialize Theorem* (2.1) *to the case when A is a classical Cartan matrix of finite type, we recover Kostant's one of the main theorems* [16, *Theorem* 5.7].

Now, we come to the proof of the theorem. Throughout the proof, we write L for $L(\lambda_0)$. Fix an orthonormal (with respect to the positive definite Hermitian form) basis $\{y_\varphi\}_{\varphi\in I}$ (respectively $\{v_a\}_{a\in I'}$) of $\mathbf{u}^-$ (resp. L), consisting of weight vectors, and define $x_\varphi = -\omega_0(y_\varphi)$. Clearly $\{x_\varphi\}_{\varphi\in I}$ is an orthonormal basis of $\mathbf{u}$. Further, $\langle x_\varphi, y_\phi\rangle = \delta_{\varphi,\phi}$ (the Kronecker delta). Choose any $\mathbf{C}$-basis $\{h_m\}_{m\in J}$ of $\mathbf{r}$, consisting of weight vectors, and let $\{h_m^\#\}_{m\in J}$ be the dual basis, with respect to $\langle\ ,\ \rangle$, of $\mathbf{r}$, i.e., $\langle h_m, h_n^\#\rangle = \delta_{m,n}$ for $m, n \in J$. Of course J is finite. *Throughout; the symbols* φ, ϕ, γ (*resp.* m, n) (*resp.* a, b) *would be assumed to run over I* (*resp. J*) (*resp.* I'). For later purposes, we well order I.

Since $\mathbf{q} = \mathbf{u}^- \oplus \mathbf{p}$ (see §1.3), $\mathbf{u}^-$ is canonically isomorphic (as $\mathbf{r}$-module) with $\mathbf{q}/\mathbf{p}$. So, the adjoint $\partial^*\colon \Lambda(\mathbf{u}^-) \otimes L \to \Lambda(\mathbf{u}^-) \otimes L$ can be thought of as a map (again denoted by) $\partial^*\colon \Lambda(\mathbf{q}/\mathbf{p}) \otimes L \to \Lambda(\mathbf{q}/\mathbf{p}) \otimes L$.

We have the following (observe that $\mathbf{q}/\mathbf{p}$, and hence $\Lambda(\mathbf{q}/\mathbf{p})$, is a $\mathbf{p}$-module under the adjoint action).

2.4 Lemma (*Expression for* ∂^*). *For* $Y \in \Lambda(\mathbf{u}^-)$ *and* $v \in L$,

$$\partial^*(Y \otimes v) = -\sum_{\varphi} y_\varphi \wedge Y \otimes x_\varphi v - \tfrac{1}{2}\sum_{\varphi} y_\varphi \wedge (\operatorname{ad} x_\varphi Y) \otimes v.$$

(*Since* $(\operatorname{ad} x_\varphi)Y$ *and* $x_\varphi v$ *are zero for all but finitely many* φ'*s, the above expression makes sense.*)

Proof. Define two operators ∂_1 and ∂_2 (of degree -1): $\Lambda(\mathbf{u}^-) \otimes L \to \Lambda(\mathbf{u}^-) \otimes L$, by

$$\partial_1(y_1 \wedge \cdots \wedge y_s \otimes v) = \partial_{\mathbf{u}^-}(y_1 \wedge \cdots \wedge y_s) \otimes v$$

(where $\partial_{\mathbf{u}^-}$ is the differential of the chain complex $\Lambda(\mathbf{u}^-, \mathbf{C})$) and

$$\partial_2(y_1 \wedge \cdots \wedge y_s \otimes v) = \sum_{p=1}^{s} (-1)^p (y_1 \wedge \cdots \wedge \hat{y}_p \wedge \cdots \wedge y_s) \otimes y_p v,$$

for $y_1, \cdots, y_s \in \mathbf{u}^-$ and $v \in L$.

By definition $\partial = \partial_1 + \partial_2$. Clearly,

$$\begin{aligned}
(\partial_{\mathbf{u}^-})^* y &= -\sum_{\varphi < \phi} \{y, [y_\varphi, y_\phi]\} y_\varphi \wedge y_\phi \quad \text{for } y \in \mathbf{u}^- \\
&= -\tfrac{1}{2}\sum_{\varphi,\phi} \{y, [y_\varphi, y_\phi]\} y_\varphi \wedge y_\phi \\
&= \tfrac{1}{2}\sum_{\varphi,\phi} \{[x_\phi, y], y_\varphi\} y_\varphi \wedge y_\phi \quad (\text{from } \mathbf{q}\text{-invariance of } \langle\ ,\ \rangle) \\
&= \tfrac{1}{2}\sum_{\phi} \operatorname{ad} x_\phi(y) \wedge y_\phi = -\tfrac{1}{2}\sum_{\phi} y_\phi \wedge \operatorname{ad} x_\phi(y).
\end{aligned}$$

Consider the operator θ: $\Lambda(\mathbf{q}/\mathbf{p}) \to \Lambda(\mathbf{q}/\mathbf{p})$, defined by

$$\theta(Y) = \sum_{\phi} y_\phi \wedge \operatorname{ad} x_\phi(Y), \text{ for } Y \in \Lambda(\mathbf{q}/\mathbf{p}).$$

It is easy to see that θ is an antiderivation (of degree $+1$). Also, $(\partial_{\mathbf{u}^-})^*$ can be seen to be an antiderivation: $\Lambda(\mathbf{q}/\mathbf{p}) \to \Lambda(\mathbf{q}/\mathbf{p})$. Hence $(\partial_{\mathbf{u}^-})^* Y = -\tfrac{1}{2}\Sigma_\phi\, y_\phi \wedge \operatorname{ad} x_\phi(Y)$ for all $Y \in \Lambda(\mathbf{q}/\mathbf{p})$. So we have

$$(\mathrm{I}_1) \quad \partial_1^*(Y \otimes v) = -\tfrac{1}{2}\sum_{\varphi} y_\varphi \wedge \operatorname{ad} x_\varphi(Y) \otimes v \quad \text{for } Y \in \Lambda(\mathbf{q}/\mathbf{p}) \text{ and } v \in L.$$

Now, we seek an expression for ∂_2^*.

$$\partial_2^*(y_1 \wedge \cdots \wedge y_s \otimes v)$$
$$= \sum_{\substack{a \in I' \\ \varphi_1 < \cdots < \varphi_{s+1}}} \left\{ y_1 \wedge \cdots \wedge y_s \otimes v, \partial_2\left(y_{\varphi_1} \wedge \cdots \wedge y_{\varphi_{s+1}} \otimes v_a\right)\right\} y_{\varphi_1} \wedge \cdots \wedge y_{\varphi_{s+1}} \otimes v_a$$
$$= \frac{1}{(s+1)!} \sum_{\substack{a \in I' \\ \varphi_1, \dots, \varphi_{s+1} \in I}} \left\{ y_1 \wedge \cdots \wedge y_s \otimes v, \partial_2\left(y_{\varphi_1} \wedge \cdots \wedge y_{\varphi_{s+1}} \otimes v_a\right)\right\} y_{\varphi_1} \wedge \cdots \wedge y_{\varphi_{s+1}} \otimes v_a$$
$$= \frac{1}{(s+1)!} \sum_{\substack{a \in I' \\ \varphi_1, \dots, \varphi_{s+1} \in I}} \sum_{p=1}^{s+1} (-1)^p \left\{ y_1 \wedge \cdots \wedge y_s, y_{\varphi_1} \wedge \cdots \wedge \hat{y}_{\varphi_p} \wedge \cdots \wedge y_{\varphi_{s+1}} \right\} \left\{ v, y_{\varphi_p} \cdot v_a \right\} y_{\varphi_1} \wedge \cdots \wedge y_{\varphi_{s+1}} \otimes v_a$$
$$= \frac{(-1)}{(s+1)!} \sum_{p=1}^{s+1} \sum_{\substack{a \in I' \\ \varphi_1, \dots, \varphi_{s+1} \in I}} \left\{ y_1 \wedge \cdots \wedge y_s, y_{\varphi_1} \wedge \cdots \wedge \hat{y}_{\varphi_p} \wedge \cdots \wedge y_{\varphi_{s+1}} \right\} y_{\varphi_p} \wedge y_{\varphi_1} \wedge \cdots \wedge \hat{y}_{\varphi_p} \wedge \cdots \wedge y_{\varphi_{s+1}} \otimes \left\{ x_{\varphi_p} v, v_a \right\} v_a$$

(from the **k** invariance of $\{\ ,\ \}$ on L)

$$= -\frac{s!}{(s+1)!} \sum_{p=1}^{s+1} \sum_{\varphi_p \in I} y_{\varphi_p} \wedge y_1 \wedge \cdots \wedge y_s \otimes x_{\varphi_p} v.$$

So

$$(\mathrm{I}_2) \quad \partial_2^*(y_1 \wedge \cdots \wedge y_s \otimes v) = -\sum_{\varphi \in I} y_\varphi \wedge y_1 \wedge \cdots \wedge y_s \otimes x_\varphi \cdot v$$

for $y_1 \wedge \cdots \wedge y_s \in \Lambda^s(\mathbf{u}^-)$ and $v \in L$.

Adding (I_1) and (I_2) we get the lemma.

We extend the operator $\partial^*\colon \Lambda(\mathbf{q}/\mathbf{p}) \otimes L \to \Lambda(\mathbf{q}/\mathbf{p}) \otimes L$ to the operator $\mathrm{Id} \otimes \partial^*\colon U(\mathbf{u}^-) \otimes_{\mathbf{C}} [\Lambda(\mathbf{q}/\mathbf{p}) \otimes L] \to U(\mathbf{u}^-) \otimes_{\mathbf{C}} [\Lambda(\mathbf{q}/\mathbf{p}) \otimes L]$. Since the canonical map $U(\mathbf{u}^-) \otimes_{\mathbf{C}} [\Lambda(\mathbf{q}/\mathbf{p}) \otimes L] \to U(\mathbf{q}) \otimes_{U(\mathbf{p})} [\Lambda(\mathbf{q}/\mathbf{p}) \otimes L]$ is an isomorphism (the **p**-module structure on $\Lambda(\mathbf{q}/\mathbf{p}) \otimes L$ is just the tensor product module structure), we get an operator (of degree $+1$)

$$\tilde{\partial}^*\colon U(\mathbf{q}) \otimes_{U(\mathbf{p})} [\Lambda(\mathbf{q}/\mathbf{p}) \otimes L] \to U(\mathbf{q}) \otimes_{U(\mathbf{p})} [\Lambda(\mathbf{q}/\mathbf{p}) \otimes L].$$

Caution. $\tilde{\partial}^*$ is, in general, not a $U(\mathbf{q})$-module map, but it is indeed a $U(\mathbf{u}^-)$-module map.

We further describe a differential $\tilde{\partial}$ of degree -1 on $U(\mathbf{q}) \otimes_{U(\mathbf{p})}[\Lambda(\mathbf{q}/\mathbf{p}) \otimes L]$.

For any Lie algebra pair $(\mathbf{b}, \mathbf{a})$, there is a standard $(\mathbf{b}, \mathbf{a})$ free resolution of the trivial one-dimensional $\mathbf{b}$-module $\mathbf{C}$ as follows (see [2, §9]).

$$\cdots \to U(\mathbf{b}) \otimes_{U(\mathbf{a})} \Lambda^s(\mathbf{b}/\mathbf{a}) \xrightarrow{\partial_s} \cdots \to U(\mathbf{b}) \otimes_{U(\mathbf{a})} \Lambda^1(\mathbf{b}/\mathbf{a})$$
$$\xrightarrow{\partial_1} U(\mathbf{b}) \otimes_{U(\mathbf{a})} \Lambda^0(\mathbf{b}/\mathbf{a}) \xrightarrow{\varepsilon_0} \mathbf{C} \to 0.$$

On tensoring this resolution with any $\mathbf{b}$-module V, we get a resolution (of the $\mathbf{b}$-module V)

$$\cdots \to \left[U(\mathbf{b}) \otimes_{U(\mathbf{a})} \Lambda^s(\mathbf{b}/\mathbf{a})\right] \otimes_{\mathbf{C}} V \xrightarrow{\partial_s \otimes \mathrm{Id}} \cdots$$
$$\to \left[U(\mathbf{b}) \otimes_{U(\mathbf{a})} \Lambda^0(\mathbf{b}/\mathbf{a})\right] \otimes_{\mathbf{C}} V \xrightarrow{\varepsilon_0 \otimes \mathrm{Id}} V \to 0.$$

Now, by using the Hopf algebra principle as given in [6, Proposition 1.7], we can identify (as $\mathbf{b}$-modules)

$$U(\mathbf{b}) \otimes_{U(\mathbf{a})} [\Lambda^s(\mathbf{b}/\mathbf{a}) \otimes_{\mathbf{C}} V] \overset{\psi}{\tilde{\to}} \left[U(\mathbf{b}) \otimes_{U(\mathbf{a})} \Lambda^s(\mathbf{b}/\mathbf{a})\right] \otimes_{\mathbf{C}} V.$$

(We will describe ψ in the next lemma.)

Define a differential $\tilde{\partial}_s$ (abbreviated to $\tilde{\partial}$ in the sequel): $U(\mathbf{q}) \otimes_{U(\mathbf{p})}[\Lambda^s(\mathbf{q}/\mathbf{p}) \otimes L] \to U(\mathbf{q}) \otimes_{U(\mathbf{p})}[\Lambda^{s-1}(\mathbf{q}/\mathbf{p}) \otimes L]$, by $\tilde{\partial}_s = \psi^{-1}(\partial_s \otimes \mathrm{Id})\psi$. Of course, $\tilde{\partial}$ is a $\mathbf{q}$-module map. (Here we have substituted $\mathbf{q}$, $\mathbf{p}$ and L for $\mathbf{b}$, $\mathbf{a}$ and V respectively.) The following lemma describes $\tilde{\partial}$.

2.5 Lemma.

$$\tilde{\partial}(A \otimes (Y \otimes v)) = A \otimes (\partial_{\mathbf{u}^-}(Y) \otimes v)$$
$$+ \sum_{p=1}^{s} \left[(-1)^{p+1} A y_p \otimes (Y^{(p)} \otimes v) + (-1)^p A \otimes \left(Y^{(p)} \otimes y_p v\right)\right]$$

for $A \in U(\mathbf{q})$, $Y = y_1 \wedge \cdots \wedge y_s \in \Lambda^s(\mathbf{u}^-)$ *and* $v \in L$.

Here $Y^{(p)}$ denotes $y_1 \wedge \cdots \wedge \hat{y}_p \wedge \cdots \wedge y_s$ and $\partial_{\mathbf{u}^-}(Y)$ is the differential of Y in the chain complex $\Lambda(\mathbf{u}^-, \mathbf{C})$.

Proof. Let us recall the expression for ψ and ψ^{-1} from [6, Proposition 1.7]:

$$\psi(A \otimes (Y \otimes v)) = \sum_n (A_{1n} \otimes Y) \otimes A_{2n} v,$$
$$\psi^{-1}((A \otimes Y) \otimes v) = \sum_n A_{1n} \otimes (Y \otimes T(A_{2n}) \cdot v)$$

for $A \in U(\mathbf{q})$, $Y \in \Lambda(\mathbf{q}/\mathbf{p})$ and $v \in L$. Here $\Delta(A) = \Sigma_n A_{1n} \otimes A_{2n}$ is the diagonal map: $U(\mathbf{q}) \to U(\mathbf{q}) \otimes U(\mathbf{q})$ and T is the unique anti-automorphism: $U(\mathbf{q}) \to U(\mathbf{q})$, which is -1 on $\mathbf{q}$.

Using these expressions for ψ and ψ^{-1}, the lemma is easy to prove. q.e.d.

Define $\tilde{\Delta} = \tilde{\partial}\tilde{\partial}^* + \tilde{\partial}^*\tilde{\partial}$. $\tilde{\Delta}$ is an operator of degree 0 on $U(\mathbf{q}) \otimes_{U(\mathbf{p})}[\Lambda(\mathbf{q}/\mathbf{p}) \otimes L]$. Now, using Lemmas (2.4) and (2.5), we get the following expression (all the expressions are in the space $U(\mathbf{q}) \otimes_{U(\mathbf{p})}[\Lambda(\mathbf{q}/\mathbf{p}) \otimes L]$):

$$2\tilde{\Delta}(1 \otimes Y \otimes v) = \tilde{\partial}\Big(-\sum_{\varphi} 2 \otimes y_\varphi \wedge Y \otimes x_\varphi v - \sum_{\varphi} 1 \otimes y_\varphi \wedge (\operatorname{ad} x_\varphi) Y \otimes v\Big)$$
$$+2\tilde{\partial}^*\Big(1 \otimes \partial_{\mathbf{u}^-}(Y) \otimes v + \sum_{p=1}^{s} (-1)^{p+1} y_p \otimes Y^{(p)} \otimes v$$
$$+ \sum_{p=1}^{s} (-1)^p \otimes Y^{(p)} \otimes y_p v\Big),$$

for $Y = y_1 \wedge \cdots \wedge y_s \in \Lambda^s(\mathbf{u}^-)$ and $v \in L$. ($Y^{(p)}$ is defined in the statement of Lemma (2.5)).

Again applying Lemmas (2.4) and (2.5) and cancelling, we get the following lemma.

2.6 Lemma. *For $Y = y_1 \wedge \cdots \wedge y_s \in \Lambda^s(\mathbf{u}^-)$ and $v \in L$, we have*

$$2\tilde{\Delta}(1 \otimes Y \otimes v) = \sum_{\varphi \in I} \Big[2 \otimes (\operatorname{ad} y_\varphi) Y \otimes x_\varphi v - 1 \otimes \partial_{\mathbf{u}^-}\big(y_\varphi \wedge (\operatorname{ad} x_\varphi) Y\big) \otimes v$$
$$-1 \otimes y_\varphi \wedge \operatorname{ad} x_\varphi(\partial_{\mathbf{u}^-} Y) \otimes v$$
$$-2 y_\varphi \otimes Y \otimes x_\varphi v + 2 \otimes Y \otimes y_\varphi x_\varphi v$$
$$+ \sum_{p=1}^{s} (-1)^p 2 \otimes y_\varphi \wedge Y^{(p)} \otimes [y_p, x_\varphi] v$$
$$+ \sum_{p=1}^{s} (-1)^p \otimes y_\varphi \wedge Y^{(p)} \otimes [x_\varphi, y_p]_- v$$
$$+ \sum_{p=1}^{s} (-1)^{p+1} [x_\varphi, y_p]_- \otimes y_\varphi \wedge Y^{(p)} \otimes v$$
$$-y_\varphi \otimes (\operatorname{ad} x_\varphi) Y \otimes v + 1 \otimes (\operatorname{ad} x_\varphi) Y \otimes y_\varphi v\Big],$$

where $[x_\varphi, y_p]_-$ *denotes the image of* $[x_\varphi, y_p]$ *under the canonical projection* $\mathbf{q} = \mathbf{u}^- \oplus \mathbf{p} \to \mathbf{u}^-$.

Proof. The lemma can be easily checked, using the well-known (and easy to verify) relation

$$(\mathrm{I}_3) \quad \partial_{\mathbf{u}^-}(y \wedge Y) + y \wedge \partial_{\mathbf{u}^-}(Y) = -(\operatorname{ad} y)Y \quad \text{for } y \in \mathbf{u}^- \text{ and } Y \in \Lambda(\mathbf{u}^-).$$

q.e.d.

Let us specialize the above lemma to the case when L is the one-dimensional trivial module $\mathbf{C}$. Let us denote the Laplacian in this case by $\tilde{\Delta}_0$. Then we have

$$(\mathrm{I}_4) \quad \begin{aligned} 2\tilde{\Delta}_0(1 \otimes Y) = \sum_{\varphi} \Big[&-1 \otimes \partial_{\mathbf{u}^-}\big(y_\varphi \wedge \operatorname{ad} x_\varphi(Y)\big) - y_\varphi \otimes (\operatorname{ad} x_\varphi)Y \\ &+ \sum_{p=1}^{s} (-1)^{p+1}[x_\varphi, y_p]_- \otimes y_\varphi \wedge Y^{(p)} \\ &-1 \otimes y_\varphi \wedge (\operatorname{ad} x_\varphi)\partial_{\mathbf{u}^-}(Y)\Big] \end{aligned}$$

for $Y = y_1 \wedge \cdots \wedge y_s \in \Lambda^s(\mathbf{u}^-)$.

We have the following, slightly simpler, expression for $\tilde{\Delta}_0$.

2.7 Lemma.

$$\begin{aligned} 2\tilde{\Delta}_0(1 \otimes Y) = &\sum_{\varphi} (-2) y_\varphi \otimes \operatorname{ad} x_\varphi(Y) + \sum_{p=1}^{s} (-1)^{p+1} \otimes F(y_p) \wedge Y^{(p)} \\ &+ \sum_{m \in J} \sum_{p=1}^{s} (-1)^p \otimes \operatorname{ad} h_m(y_p) \wedge (\operatorname{ad} h_m^{\#}) Y^{(p)}, \end{aligned}$$

(in $U(\mathbf{q}) \otimes_{U(\mathbf{p})} \Lambda(\mathbf{q}/\mathbf{p})$) for $Y = y_1 \wedge \cdots \wedge y_s \in \Lambda^s(\mathbf{u}^-)$. ($\{h_m\}$ and $\{h_m^{\#}\}_{m \in J}$ are defined just before Lemma (2.4)). The operator $F = F_S \colon \mathbf{u}^- \to \mathbf{u}^-$ is defined by $F(y) = \sum_{\varphi \in I} [y_\varphi, [x_\varphi, y]_-]$ for $y \in \mathbf{u}^-$.

Remark. We would describe F more explicitly in Proposition (2.10).

Proof.

$$\begin{aligned} &\sum_{\varphi} \sum_{p=1}^{s} (-1)^{p+1}[x_\varphi, y_p]_- \otimes y_\varphi \wedge Y^{(p)} \\ &\quad = \sum_{\varphi} \sum_{p} \sum_{\phi} (-1)^{p+1}\{[x_\varphi, y_p], y_\phi\} y_\phi \otimes y_\varphi \wedge Y^{(p)} \\ &\quad = \sum_{\phi} \sum_{p} \sum_{\varphi} (-1)^{p+1} y_\phi \otimes \{[x_\varphi, y_p], y_\phi\} y_\varphi \wedge Y^{(p)} \\ &\quad = \sum_{\phi} \sum_{p=1}^{s} (-1)^{p+1} y_\phi \otimes [y_p, x_\phi]_- \wedge Y^{(p)} \quad \text{(by the invariance of } \{\ ,\ \}\text{)}, \end{aligned}$$

$$(\mathrm{I}_5) \quad \sum_{\varphi} \sum_{p=1}^{s} (-1)^{p+1}[x_\varphi, y_p]_- \otimes y_\varphi \wedge Y^{(p)} = -\sum_{\varphi} y_\varphi \otimes (\operatorname{ad} x_\varphi)Y.$$

Further,

$$\sum_{\varphi} 1 \otimes \operatorname{ad} y_{\varphi}(\operatorname{ad} x_{\varphi} Y)$$

$\big((\operatorname{ad} x_{\varphi})Y$ is to be interpreted as an element of $\Lambda(\mathbf{u}^-) \approx \Lambda(\mathbf{q}/\mathbf{p})\big)$.

$$= \sum_{\varphi} \sum_{p} (-1)^{p-1} \otimes \operatorname{ad} y_{\varphi}\big([x_{\varphi}, y_p]_- \wedge Y^{(p)}\big)$$

$$= \sum_{\varphi} \sum_{p} (-1)^{p-1} \otimes \big[y_{\varphi}, [x_{\varphi}, y_p]_-\big] \wedge Y^{(p)}$$

$$+ \sum_{\varphi} \sum_{p} (-1)^{p-1} \otimes [x_{\varphi}, y_p]_- \wedge \operatorname{ad} y_{\varphi}(Y^{(p)}).$$

So, using the definition of F, we get

$$(\mathrm{I}_6) \qquad \sum_{\varphi} 1 \otimes \operatorname{ad} y_{\varphi}(\operatorname{ad} x_{\varphi} Y) = \sum_{p=1}^{s} (-1)^{p-1} \otimes F(y_p) \wedge Y^{(p)} + \sum_{\varphi} \sum_{p} (-1)^{p-1} \otimes [x_{\varphi}, y_p]_- \wedge \operatorname{ad} y_{\varphi}(Y^{(p)}).$$

Now we seek an expression for $\partial_{\mathbf{u}^-}(\operatorname{ad} x_{\varphi} Y) - \operatorname{ad} x_{\varphi}(\partial_{\mathbf{u}^-}(Y))$. We have described an operator $\tilde{\partial}$ (of degree -1) on $U(\mathbf{q}) \otimes_{U(\mathbf{p})} [\Lambda(\mathbf{q}/\mathbf{p}) \otimes L]$, just before Lemma (2.5). Let us specialize to the case when L is the one-dimensional trivial module $\mathbf{C}$ and denote the operator $\tilde{\partial}$, in this case, by $\tilde{\partial}_0$. Since $\tilde{\partial}_0$ is a $U(\mathbf{q})$-module map,

$$\tilde{\partial}_0\big(1 \otimes \operatorname{ad} x_{\varphi}(Y)\big) = x_{\varphi} \cdot \tilde{\partial}_0(1 \otimes Y)$$

$$= x_{\varphi}\Big[1 \otimes \partial_{\mathbf{u}^-}(Y) + \sum_{p} (-1)^{p+1} y_p \otimes Y^{(p)}\Big],$$

(by Lemma (2.5)).

Further, it is easy to verify that the following diagram is commutative:

$$\begin{array}{ccc} \mathbf{C} \underset{U(\mathbf{u}^-)}{\otimes} \Big[U(\mathbf{q}) \underset{U(\mathbf{p})}{\otimes} \Lambda(\mathbf{q}/\mathbf{p})\Big] & \xrightarrow{\mathrm{Id} \otimes \tilde{\partial}_0} & \mathbf{C} \underset{U(\mathbf{u}^-)}{\otimes} \Big[U(\mathbf{q}) \underset{U(\mathbf{p})}{\otimes} \Lambda(\mathbf{q}/\mathbf{p})\Big] \\ \uparrow \wr & & \uparrow \wr \\ \Lambda(\mathbf{q}/\mathbf{p}) & \xrightarrow[\partial_{\mathbf{u}^-}]{} & \Lambda(\mathbf{q}/\mathbf{p}) \end{array}$$

(The vertical map is $Y \mapsto 1 \otimes (1 \otimes Y)$.) Hence

$$(\mathrm{I}_7) \qquad \partial_{\mathbf{u}^-}\big(\operatorname{ad} x_{\varphi}(Y)\big) = \sum_{p=1}^{s} (-1)^{p+1} \operatorname{ad}\big([x_{\varphi}, y_p] - [x_{\varphi}, y_p]_-\big) Y^{(p)} + \operatorname{ad} x_{\varphi}\big(\partial_{\mathbf{u}^-}(Y)\big) \text{ in } \Lambda(\mathbf{u}^-).$$

Finally,

$$\sum_{\varphi} y_\varphi \wedge \operatorname{ad}\big([x_\varphi, y_p] - [x_\varphi, y_p]_-\big)Y^{(p)}$$

$$= \sum_{\varphi}\sum_{m} y_\varphi \wedge \big\langle [x_\varphi, y_p], h_m\big\rangle \operatorname{ad} h_m^{\#}(Y^{(p)})$$

$$+ \sum_{\varphi}\sum_{\phi} y_\varphi \wedge \big\{[x_\varphi, y_p], x_\phi\big\} \operatorname{ad} x_\phi(Y^{(p)})$$

$$= \sum_{m}\sum_{\varphi} \big\{[y_p, h_m], y_\varphi\big\} y_\varphi \wedge \operatorname{ad} h_m^{\#}(Y^{(p)})$$

$$+ \sum_{\phi}\sum_{\varphi} \big\{[y_p, y_\phi], y_\varphi\big\} y_\varphi \wedge \operatorname{ad} x_\phi(Y^{(p)}),$$

$$(\mathrm{I}_8)\quad \sum_{\varphi} y_\varphi \wedge \operatorname{ad}\big([x_\varphi, y_p] - [x_\varphi, y_p]_-\big)Y^{(p)} = -\sum_{m} \operatorname{ad} h_m(y_p) \wedge \operatorname{ad} h_m^{\#}(Y^{(p)}) - \sum_{\varphi} \operatorname{ad} y_\varphi(y_p) \wedge \operatorname{ad} x_\varphi(Y^{(p)}).$$

Now Lemma (2.7) follows by substituting the identities (I_3) and (I_5)–(I_8) in (I_4), along with the (trivially verified) identity

$$(\mathrm{I}_9)\quad \sum_{p=1}^{s} (-1)^{p-1} \otimes [x_\varphi, y_p]_- \wedge \operatorname{ad} y_\varphi(Y^{(p)}) + \sum_{p=1}^{s} (-1)^p \otimes \operatorname{ad} y_\varphi(y_p) \wedge \operatorname{ad} x_\varphi(Y^{(p)}) = 0.$$

q.e.d.

We have two expressions for $2\tilde{\Delta}_0(1 \otimes Y)$, one given by the identity (I_4) and the other by Lemma (2.7). Equating these two expressions, we get the identity

$$\sum_{\varphi} \Big[-1 \otimes \partial_{\mathbf{u}^-}\big(y_\varphi \wedge \operatorname{ad} x_\varphi(Y)\big) - y_\varphi \otimes \operatorname{ad} x_\varphi(Y) + \sum_{p=1}^{s} (-1)^{p+1}[x_\varphi, y_p]_- \otimes y_\varphi \wedge Y^{(p)} - 1 \otimes y_\varphi \wedge \operatorname{ad} x_\varphi(\partial_{\mathbf{u}^-} Y)\Big] \otimes v$$

$$(\mathrm{I}_{10})\quad = \Bigg[\sum_{\varphi} (-2) y_\varphi \otimes \operatorname{ad} x_\varphi(Y) + \sum_{p=1}^{s} (-1)^{p+1} \otimes F(y_p) \wedge Y^{(p)} + \sum_{m\in J}\sum_{p=1}^{s} (-1)^p \otimes \operatorname{ad} h_m(y_p) \wedge \operatorname{ad} h_m^{\#}(Y^{(p)})\Bigg] \otimes v$$

valid in $[U(\mathbf{q}) \otimes_{U(\mathbf{p})} \Lambda(\mathbf{g}/\mathbf{p})] \otimes_{\mathbf{C}} L$, where $Y = y_1 \wedge \cdots \wedge y_s \in \Lambda^s(\mathbf{u}^-)$ and $v \in L$.

Taking ψ^{-1} (see the proof of Lemma (2.5)) of both the sides, we get the identity

$$(\mathrm{I}_{11}) \qquad I_{11} = \psi^{-1}(I_{10}) \quad \text{valid in } U(\mathbf{q}) \underset{U(\mathbf{p})}{\otimes} [\Lambda(\mathbf{q}/\mathbf{p}) \otimes L].$$

We prove the following lemma, which would be used to simplify the expression for $\tilde{\Delta}$.

2.8 Lemma.

$$\sum_{\varphi} \Bigg[\sum_{p=1}^{s} (-1)^p \otimes y_\varphi \wedge Y^{(p)} \otimes [y_p, x_\varphi] v + 1 \otimes (\operatorname{ad} y_\varphi) Y \otimes x_\varphi v + 1 \otimes (\operatorname{ad} x_\varphi) Y \otimes y_\varphi v \Bigg] = -\sum_m 1 \otimes \operatorname{ad} h_m(Y) \otimes h_m^{\#} v, \quad \text{in } U(\mathbf{q}) \underset{U(\mathbf{p})}{\otimes} [\Lambda(\mathbf{q}/\mathbf{p}) \otimes L],$$

for any $Y = y_1 \wedge \cdots \wedge y_s \in \Lambda^s(\mathbf{u}^-)$ *and* $v \in L$.

Proof.

$$\begin{aligned}
\sum_\varphi \sum_p (-1)^p \otimes y_\varphi \wedge Y^{(p)} \otimes [y_p, x_\varphi] v
&= \sum_\varphi \sum_p \sum_\phi (-1)^p \otimes y_\varphi \wedge Y^{(p)} \otimes \{[y_p, x_\varphi], x_\phi\} x_\phi v \\
&\quad + \sum_\varphi \sum_p \sum_\phi (-1)^p \otimes y_\varphi \wedge Y^{(p)} \otimes \{[y_p, x_\varphi], y_\phi\} y_\phi v \\
&\quad + \sum_\varphi \sum_p \sum_m (-1)^p \otimes y_\varphi \wedge Y^{(p)} \otimes \langle [y_p, x_\varphi], h_m \rangle h_m^{\#} v \\
&= \sum_\phi \sum_p \sum_\varphi (-1)^{p+1} \otimes \{[y_p, y_\phi], y_\varphi\} y_\varphi \wedge Y^{(p)} \otimes x_\phi v \\
&\quad + \sum_\phi \sum_p \sum_\varphi (-1)^{p+1} \otimes \{[y_p, x_\phi], y_\varphi\} y_\varphi \wedge Y^{(p)} \otimes y_\phi v \\
&\quad + \sum_m \sum_p \sum_\varphi (-1)^{p+1} \otimes \{[y_p, h_m], y_\varphi\} y_\varphi \wedge Y^{(p)} \otimes h_m^{\#} v \\
&= \sum_\phi \left[-1 \otimes \operatorname{ad} y_\phi(Y) \otimes x_\phi v - 1 \otimes \operatorname{ad} x_\phi(Y) \otimes y_\phi v \right] \\
&\quad - \sum_m 1 \otimes \operatorname{ad} h_m(Y) \otimes h_m^{\#} v.
\end{aligned}$$

Now, the lemma follows.

Substituting the identity (I_{11}) in the expression of $2\tilde{\Delta}$ (as given in Lemma (2.6)) and using Lemma (2.8), we get (after cancellation)

$$2\tilde{\Delta}(1 \otimes Y \otimes v) = \sum_{\varphi} \left[-2y_\varphi \otimes Y \otimes x_\varphi v - 2y_\varphi \otimes \operatorname{ad} x_\varphi(Y) \otimes v\right]$$

$$(I_{12}) \qquad + \sum_{\varphi} 2 \otimes Y \otimes y_\varphi x_\varphi v + \sum_{p=1}^{s} (-1)^{p+1} \otimes F(y_p) \wedge Y^{(p)} \otimes v$$

$$+ \sum_{m} \sum_{p=1}^{s} (-1)^{p} \otimes \operatorname{ad} h_m(y_p) \wedge \operatorname{ad} h_m^{\#}(Y^{(p)}) \otimes v$$

$$- \sum_{m} 2 \otimes \operatorname{ad} h_m(Y) \otimes h_m^{\#} v$$

for $Y = y_1 \wedge \cdots \wedge y_s \in \Lambda^s(\mathfrak{u}^-)$ and $v \in L$.

The following lemma, which relates $\tilde{\Delta}$ and Δ, is easy to prove.

2.9 Lemma. $\tilde{\Delta}$: $U(\mathfrak{q}) \otimes_{U(\mathfrak{p})}[\Lambda(\mathfrak{q}/\mathfrak{p}) \otimes_{\mathbf{C}} L] \to U(\mathfrak{q}) \otimes_{U(\mathfrak{p})}[\Lambda(\mathfrak{q}/\mathfrak{p}) \otimes_{\mathbf{C}} L]$ *is a* $U(\mathfrak{u}^-)$*-module map and the following diagram is commutative.*

$$\begin{array}{ccc} \Lambda(\mathfrak{q}/\mathfrak{p}) \underset{\mathbf{C}}{\otimes} L & \xrightarrow{\Delta} & \Lambda(\mathfrak{q}/\mathfrak{p}) \underset{\mathbf{C}}{\otimes} L \\ \downarrow & & \downarrow \\ \mathbf{C} \underset{U(\mathfrak{u}^-)}{\otimes} \left[U(\mathfrak{q}) \underset{U(\mathfrak{p})}{\otimes} [\Lambda(\mathfrak{q}/\mathfrak{p}) \otimes L]\right] & \xrightarrow[\mathrm{Id} \otimes \tilde{\Delta}]{} & \mathbf{C} \underset{U(\mathfrak{u}^-)}{\otimes} \left[U(\mathfrak{q}) \underset{U(\mathfrak{p})}{\otimes} \left[\Lambda(\mathfrak{q}/\mathfrak{p}) \underset{\mathbf{C}}{\otimes} L\right]\right] \end{array}$$

Here the vertical map is the canonical map given by

$$Y \otimes v \mapsto 1 \otimes (1 \otimes (Y \otimes v)).$$

Assuming the next proposition, we are in position to complete the proof of Theorem (2.1).

Using the above Lemma (2.9) and the identity (I_{12}), we get

(I_{13})

$$2\Delta(Y \otimes v)$$

$$= \sum_{\varphi} 2Y \otimes y_\varphi x_\varphi v + \sum_{p=1}^{s} (-1)^{p+1} F(y_p) \wedge Y^{(p)} \otimes v$$

$$+ \sum_{m} \left[\sum_{p=1}^{s} (-1)^{p} \operatorname{ad} h_m(y_p) \wedge \operatorname{ad} h_m^{\#}(Y^{(p)}) \otimes v - 2 \operatorname{ad} h_m(Y) \otimes h_m^{\#} v\right]$$

(continues)

$$= \sum_{\varphi} 2Y \otimes y_\varphi x_\varphi v + \sum_{p=1}^{s} \sum_{m} (-1)^p \operatorname{ad} h_m^{\#} \operatorname{ad} h_m(y_p) \wedge Y^{(p)} \otimes v$$

$$+ \sum_{p=1}^{s} \sum_{m} (-1)^p \operatorname{ad} h_m(y_p) \wedge \operatorname{ad} h_m^{\#}(Y^{(p)}) \otimes v - \sum_{m} \operatorname{ad} h_m(Y) \otimes h_m^{\#} v$$

$$- \sum_{m} \operatorname{ad} h_m^{\#}(Y) \otimes h_m v - \sum_{m} \operatorname{ad} h_m^{\#} \operatorname{ad} h_m(Y) \otimes v - \sum_{m} Y \otimes h_m^{\#} h_m v$$

$$+ \sum_{m} \operatorname{ad} h_m^{\#} \operatorname{ad} h_m(Y) \otimes v + \sum_{m} Y \otimes h_m^{\#} h_m v.$$

(By the next Proposition (2.10) and using $\Sigma_m \operatorname{ad} h_m(Y) \otimes h_m^{\#} v = \Sigma_m \operatorname{ad} h_m^{\#}(Y) \otimes h_m v$.)

The above sum reduces to

$$\sum_{\varphi} 2Y \otimes y_\varphi x_\varphi v - \sum_{m} \operatorname{ad} h_m^{\#} \operatorname{ad} h_m(Y) \otimes v - \sum_{m} \operatorname{ad} h_m^{\#} \operatorname{ad} h_m(Y \otimes v)$$

$$+ \sum_{m} \operatorname{ad} h_m^{\#} \operatorname{ad} h_m(Y) \otimes v + Y \otimes \sum_{m} h_m^{\#} h_m v$$

$$= Y \otimes \Gamma v - \Gamma^{\mathbf{r}}(Y \otimes v),$$

where Γ (resp. $\Gamma^{\mathbf{r}}$) is the Casimir operator for $\mathbf{q}$-modules (resp. $\mathbf{r}$-modules), as defined in §1.7, with respect to the (already fixed) nondegenerate symmetric form $\langle \, , \, \rangle$ on $\mathbf{q}$ (resp. the restricted form on $\mathbf{r}$). Here, we are using the well-known fact that $\Gamma^{\mathbf{r}} = \Sigma_{m \in J} h_m^{\#} h_m$.

Since $v \in L(\lambda_0)$, $\Gamma(v) = [\sigma(\lambda_0 + \rho, \lambda_0 + \rho) - \sigma(\rho, \rho)]v$ and for

$$\sum_{k} Y_k \otimes v_k \in W^{\beta}, \Gamma^{\mathbf{r}}\left(\sum Y_k \otimes v_k\right)$$

$$= [\sigma(\beta + \rho, \beta + \rho) - \sigma(\rho, \rho)] \sum (Y_k \otimes v_k)$$

(see §1.7, last paragraph). So the proof of the theorem is complete modulo the next proposition.

We come to the following proposition, which seems interesting on its own.

2.10 Proposition. *$F(y) = -\Sigma_{m \in J} \operatorname{ad} h_m^{\#} \operatorname{ad} h_m(y)$ for all $y \in \mathbf{u}^-$, where $F(y) = F_S(y) = \Sigma_\varphi [y_\varphi, [x_\varphi, y]_-]$ (see Lemma (2.7)).*

Proof. First, we assume the proposition for the case $S = \varnothing$ and prove it for all the subsets S of finite type.

The maps F_S and $\Sigma \operatorname{ad} h_m^{\#} \operatorname{ad} h_m$ are both $\mathbf{r}$-module maps: $\mathbf{u}^- \to \mathbf{u}^-$ (F_S is an $\mathbf{r}$-module map, can also be seen by observing that $F_S = 2\Delta_0$: $\mathbf{u}^- \to \mathbf{u}^-$, see ($\mathrm{I}_{13}$)). So, to prove the above proposition, it suffices to prove that $F_S(y) = -\Sigma_m \operatorname{ad} h_m^{\#} \operatorname{ad} h_m(y)$, for y a highest weight vector with respect to the action of $\mathbf{r}$ (with weight $-\beta$). It is easy to see that, for any such y, $F_S(y) = F_\varnothing(y)$ ($F_\varnothing$ corresponds to the case when $S = \varnothing$). But

$$F_\varnothing(y) = -[\sigma(-\beta + \rho, -\beta + \rho) - \sigma(\rho, \rho)]\, y,$$

(since we have assumed that the proposition is true in the case $S = \varnothing$). Further, $\Sigma_m \operatorname{ad} h_m^{\#} \operatorname{ad} h_m$ is the Casimir operator $\Gamma^{\mathfrak{r}}$ and hence

$$\sum_m \operatorname{ad} h_m^{\#} \operatorname{ad} h_m(y) = [\sigma(-\beta + \rho, -\beta + \rho) - \sigma(\rho, \rho)]\, y.$$

(See just before this proposition.) This completes the proof of the proposition for all the subsets S of finite type, assuming the proposition for the case $S = \varnothing$.

We come to the case $S = \varnothing$. We need to prove that

$$F_\phi(y) = -[\sigma(-\beta + \rho, -\beta + \rho) - \sigma(\rho, \rho)]\, y \quad \text{for } y \in \mathbf{q}_{-\beta} \text{ with } \beta \in \Delta_+.$$

We prove this by induction on $|\beta| = \Sigma_i m_i$, where $\beta = \Sigma m_i \alpha_i$. It is indeed true for simple roots α_i, as $\sigma(\rho - \alpha_i, \rho - \alpha_i) - \sigma(\rho, \rho) = 0$ (see §1.6).

Assume, by induction, that it is true for $y \in \mathbf{q}_{-\beta}$. We want to show that it is true for $[f_{i_0}, y]$ for any $1 \leqslant i_0 \leqslant l$. For notational convenience we take $i_0 = 1$. We can assume, without loss of generality, that $\beta \neq \alpha_1$.

For a root $\alpha = \Sigma n_i \alpha_i$, we denote $\bar{\alpha} = \Sigma_{i>1} n_i \alpha_i$ ($\bar{\alpha}$ may note be a root). By the α_1-root string through α, defined Φ_α, we mean $\{\alpha + \mathbf{Z}\alpha_1\} \cap \Delta$. Φ_α is finite [6, p. 49]. If $\alpha \in \Delta_+$ with $\bar{\alpha} \neq 0$, clearly, we have $\Phi_\alpha \subset \Delta_+$. Define the operator $f_{\Phi_\alpha}\colon \mathbf{q} \to \mathbf{q}$ by

$$f_{\Phi_\alpha}(y) = \sum_{\gamma \in \Phi_\alpha} \sum_{i=1}^{n(\gamma)} \left[y_\gamma^i, \left[x_\gamma^i, y\right]\right],$$

where $\{x_\gamma^1, \cdots, x_\gamma^{n(\gamma)}\}$ is an orthonormal basis of $\mathbf{q}_\gamma$ and $y_\gamma^i = -\omega_0(x_\gamma^i)$ for all $1 \leqslant i \leqslant n(\gamma)$. The operator f_{Φ_α} commutes with the adjoint action of the Lie-subalgebra generated by $\{e_1, f_1, h_1\}$, on $\mathbf{q}$ (see [6, proof of Proposition 4.2]). We denote $\Delta_{++} = \{\alpha \in \Delta_+ \colon \bar{\alpha} \neq 0\}$. Two roots $\alpha, \alpha' \in \Delta_{++}$ are called equivalent (denoted $\alpha \sim \alpha'$) if $\Phi_\alpha = \Phi_{\alpha'}$. Clearly $\Phi_\alpha \subset \Delta_{++}$ for $\alpha \in \Delta_{++}$. Now, we have

$$(\mathrm{I}_{14}) \qquad \begin{aligned} F_\phi(y) = {} & \sum_{\substack{\alpha \in \Delta_{++}/\sim \\ \text{with } \bar{\alpha} < \bar{\beta}}} \left[f_{\Phi_\alpha}(y)\right] + \frac{\left(1 - \delta_{m_1, 0}\right)}{\langle e_1, f_1 \rangle} [f_1, [e_1, y]] \\ & + \sum_{i=1}^{n(\beta - \alpha_1)} \left[y_{\beta - \alpha_1}^i, \left[x_{\beta - \alpha_1}^i, y\right]\right] \end{aligned}$$

for $y \in \mathbf{q}_{-\beta}$ with any $\beta = \sum m_i \alpha_i \in \Delta_{++}$, where δ is the Kronecker delta and the last term is to be interpreted as 0 if $\beta - \alpha_1 \notin \Delta_+$.

The proof of this identity is easily checked, keeping in mind:

(1) If $\alpha = \Sigma_{i=1}^l n_i \alpha_i \in \Delta$, then either all $n_i \geqslant 0$ or all $n_i \leqslant 0$, and

(2) $\mathbf{Z}\alpha_1 \cap \Delta = \{\pm \alpha_1\}$.

Applying the identity (I_{14}) to $F_\varnothing[f_1, y]$ and $F_\varnothing(y)$, we get

$$[f_1, F_\varnothing(y)] - F_\varnothing[f_1, y]$$
$$= \sum_{\substack{\alpha \in \Delta_{++}/\sim \\ \text{with } \bar{\alpha} < \bar{\beta}}} [f_1, f_{\Phi_\alpha}(y)] + \frac{(1-\delta_{m_1,0})}{\langle e_1, f_1 \rangle}[f_1,[f_1,[e_1,y]]]$$
$$+ \sum_{i=1}^{n(\beta-\alpha_1)} \left[f_1, \left[y^i_{\beta-\alpha_1}, \left[x^i_{\beta-\alpha_1}, y\right]\right]\right] - \sum_{\substack{\alpha \in \Delta_{++}/\sim \\ \text{with } \bar{\alpha} < \bar{\beta}}} f_{\Phi_\alpha}[f_1, y]$$
$$- \frac{1}{\langle e_1, f_1 \rangle}[f_1,[e_1,[f_1,y]]] - \sum_{j=1}^{n(\beta)} \left[y^j_\beta, \left[x^j_\beta, [f_1, y]\right]\right].$$

Since f_{Φ_α} commutes with the action of f_1, we have

$$(I_{15}) \qquad [f_1, F_\varnothing(y)] - F_\varnothing[f_1, y] = \frac{(1-\delta_{m_1,0})}{\langle e_1, f_1 \rangle}[f_1,[f_1,[e_1,y]]] + \sum_{i=1}^{n(\beta-\alpha_1)} \left[f_1, \left[y^i_{\beta-\alpha_1}, \left[x^i_{\beta-\alpha_1}, y\right]\right]\right] - \frac{1}{\langle e_1, f_1 \rangle}[f_1,[e_1,[f_1,y]]] - \sum_{j=1}^{n(\beta)} \left[y^j_\beta, \left[x^j_\beta, [f_1, y]\right]\right].$$

Finally, we prove the following identity.

$$(I_{16}) \qquad \sum_{i=1}^{n(\beta-\alpha_1)} \left[f_1, \left[y^i_{\beta-\alpha_1}, \left[x^i_{\beta-\alpha_1}, y\right]\right]\right] - \sum_{j=1}^{n(\beta)} \left[y^j_\beta, \left[x^j_\beta, [f_1, y]\right]\right] = \frac{1}{\langle e_1, f_1 \rangle}\left([f_1,[f_1,[e_1,y]]] - [f_1,[e_1,[f_1,y]]]\right).$$

Since $[x^i_{\beta-\alpha_1}, y] \in \mathbf{q}_{-\alpha_1}$, we have

$$\left[x^i_{\beta-\alpha_1}, y\right] = \left\{\left[x^i_{\beta-\alpha_1}, y\right], f_1\right\} \frac{f_1}{\langle e_1, f_1 \rangle} = \left\{[y, e_1], y^i_{\beta-\alpha_1}\right\} \frac{f_1}{\langle e_1, f_1 \rangle}.$$

So, substituting, we get

$$\sum_{i=1}^{n(\beta-\alpha_1)} \left[f_1, \left[y^i_{\beta-\alpha_1}, \left[x^i_{\beta-\alpha_1}, y\right]\right]\right]$$

$$= \frac{1}{\langle e_1, f_1\rangle}\left[f_1, \left[f_1, \sum_{i=1}^{n(\beta-\alpha_1)} \left\{[e_1, y], y^i_{\beta-\alpha_1}\right\} y^i_{\beta-\alpha_1}\right]\right]$$

$$= \frac{1}{\langle e_1, f_1\rangle}[f_1, [f_1, [e_1, y]]].$$

Exactly, similarly, we have

$$\sum_{j=1}^{n(\beta)} \left[y^j_\beta, \left[x^j_\beta, [f_1, y]\right]\right] = \frac{1}{\langle e_1, f_1\rangle} \sum_{j=1}^{n(\beta)} \left[\left\{\left[x^j_\beta, [f_1, y]\right], f_1\right\} y^j_\beta, f_1\right]$$

$$= \frac{1}{\langle e_1, f_1\rangle}\left[\sum_j \left\{[[f_1, y], e_1], y^j_\beta\right\} y^j_\beta, f_1\right]$$

$$= \frac{1}{\langle e_1, f_1\rangle}[[[f_1, y], e_1], f_1]$$

$$= \frac{1}{\langle e_1, f_1\rangle}[f_1, [e_1, [f_1, y]]].$$

Substituting (I_{16}) in (I_{15}), we get

$$[f_1, F_\varnothing(y)] - F_\varnothing[f_1, y]$$

$$= \frac{(2 - \delta_{m_1,0})}{\langle e_1, f_1\rangle}[f_1, [f_1, [e_1, y]]] - \frac{2}{\langle e_1, f_1\rangle}[f_1, [e_1, [f_1, y]]]$$

$$= \frac{2}{\langle e_1, f_1\rangle}([f_1, [f_1, [e_1, y]]] - [f_1, [e_1, [f_1, y]]])$$

(since, if $m_1 = 0$, $[e_1, y] = 0$)

$$(I_{17}) \qquad = \frac{-2}{\langle e_1, f_1\rangle}[f_1, [h_1, y]] = \frac{2\beta(h_1)}{\langle e_1, f_1\rangle}[f_1, y],$$

$$[f_1, F_\varnothing(y)] - F_\varnothing[f_1, y]$$

$$= -[\sigma(-\beta + \rho, -\beta + \rho) - \sigma(-\beta - \alpha_1 + \rho, -\beta - \alpha_1 + \rho)][f_1, y].$$

By induction hypothesis, $F_\varnothing(y) = -[\sigma(-\beta + \rho, -\beta + \rho) - \sigma(\rho, \rho)]y$ and hence, by (I_{17}), the proposition follows.

3. Disjointness of d and ∂.

Let $\mathbf{q} = \mathbf{q}(A)$ be the Kac-Moody Lie algebra associated to a symmetrizable generalized Cartan matrix $A = (a_{ij})_{1 \leqslant i, j \leqslant l}$ and let S be a subset of $\{1, \cdots, l\}$, of finite type. Recall the definitions of $\mathbf{u}$, $\mathbf{u}^-$, $\mathbf{r}$ and $\mathbf{p}$ from §1.3. Also recall from §1.10; $C(\mathbf{q}, \mathbf{r})$ denotes the standard co-chain complex, with differential d (of degree $+1$), associated to the Lie algebra pair $(\mathbf{q}, \mathbf{r})$ with trivial coefficients. More explicitly, $C(\mathbf{q}, \mathbf{r})$ is defined to be $\Sigma_{s \geqslant 0} \operatorname{Hom}_{\mathbf{r}}(\Lambda^s(\mathbf{q}/\mathbf{r}), \mathbf{C})$ ($\mathbf{r}$ is acting trivially on $\mathbf{C}$). Extend d to a derivation $\tilde{d}$ of $\Sigma_{s \geqslant 0} \operatorname{Hom}_{\mathbf{C}}(\Lambda^s(\mathbf{q}/\mathbf{r}), \mathbf{C})$, as follows.

Since $\mathbf{q}/\mathbf{r}$ can be canonically identified with $\mathbf{u} \oplus \mathbf{u}^-$, the projection $\mathbf{q} \to \mathbf{q}/\mathbf{r}$ has a canonical splitting θ. Define $\tilde{d}$ as the composite of

$$\sum_{s \geqslant 0} [\Lambda^s(\mathbf{q}/\mathbf{r})]^* \hookrightarrow \sum_s [\Lambda^s(\mathbf{q})]^* \xrightarrow{d} \sum_s [\Lambda^s(\mathbf{q})]^* \xrightarrow{\Omega} \sum_s [\Lambda^s(\mathbf{q}/\mathbf{r})]^*,$$

where the first map is the canonical inclusion, d is the differential of the cochain complex $\Sigma[\Lambda^s(\mathbf{q})]^*$ and Ω is the projection induced by the splitting θ. *Caution*! $\tilde{d}$, in general, is not a differential.

We have the canonical decomposition

$$[\Lambda^s(\mathbf{q}/\mathbf{r})]^* = \sum_{p+q=s} \operatorname{Hom}_{\mathbf{C}}(\Lambda^p(\mathbf{u}) \otimes \Lambda^q(\mathbf{u}^-), \mathbf{C}).$$

In the sequel, we denote $\operatorname{Hom}_{\mathbf{C}}(\Lambda^p(\mathbf{u}) \otimes \Lambda^q(\mathbf{u}^-), \mathbf{C})$ *by* $\tilde{C}^{p,q}$; $\operatorname{Hom}_{\mathbf{r}}(\Lambda^p(\mathbf{u}) \otimes \Lambda^q(\mathbf{u}^-), \mathbf{C})$ *by* $C^{p,q}$ *and put* $\tilde{C}^s = \Sigma_{p+q=s} \tilde{C}^{p,q}$; $C^s = \Sigma_{p+q=s} C^{p,q}$, *so that* $C^s = C^s(\mathbf{q}, \mathbf{r})$ *(in the earlier notation).*

It is easy to see (by using, just, the definitions) that $\tilde{d}(\tilde{C}^{p,q}) \subset \tilde{C}^{p+1,q} \oplus \tilde{C}^{p,q+1}$. Define $\tilde{d}'$: $\tilde{C}^{p,q} \to \tilde{C}^{p+1,q}$ to be the projection of $\tilde{d}$ on the first factor and $\tilde{d}''$: $\tilde{C}^{p,q} \to \tilde{C}^{p,q+1}$ to be the projection of $\tilde{d}$ on the second factor.

Similarly, we define $\tilde{\partial}'$ and $\tilde{\partial}''$ as follows. Put a Lie algebra bracket on $\mathbf{u} \oplus \mathbf{u}^-$ by demanding $[\mathbf{u}, \mathbf{u}^-] = 0$ and brackets in $\mathbf{u}$ and $\mathbf{u}^-$ are the ones coming from the brackets in $\mathbf{q}$. Let $\tilde{\partial}_0$ denote the differential (of degree -1) of the chain complex $\Lambda(\mathbf{u} \oplus \mathbf{u}^-)$. Using the nondegenerate form $\langle\ ,\ \rangle$ on $\mathbf{q}$ (described in §1.6), we canonically embed

$$(*) \qquad e = e_{\langle\ ,\ \rangle}: \sum_{s \geqslant 0} \Lambda^s(\mathbf{u} \oplus \mathbf{u}^-) \hookrightarrow \sum_{s \geqslant 0} [\Lambda^s(\mathbf{u} \oplus \mathbf{u}^-)]^*.$$

Further, we topologize $\tilde{C}^s = \operatorname{Hom}_{\mathbf{C}}(\Lambda^s(\mathbf{u} \oplus \mathbf{u}^-), \mathbf{C})$ by putting the topology of pointwise convergence, i.e., $f_\nu \to f$ in $\tilde{C}^s$ if and only if $f_\nu(v) \to f(v)$ in $\mathbf{C}$ (with the usual metric topology), for all $v \in \Lambda^s(\mathbf{u} \oplus \mathbf{u}^-)$. It is easy to see that $\tilde{C}^s$, with this topology (we would not be considering any other topology on $\tilde{C}^s$), is a complete, Hausdorff, topological vector space (see [4, Proposition 15.20]). Moreover, $e(\Lambda^s(\mathbf{u} \oplus \mathbf{u}^-))$ can be seen to be dense in $\tilde{C}^s$.

$\tilde{\partial}_0$ can be extended (uniquely), under the identification (*), to a continuous map $\tilde{\partial}: \tilde{C}^s \to \tilde{C}^{s-1}$. Further, $\tilde{\partial}(\tilde{C}^{p,q}) \subset \tilde{C}^{p-1,q} \oplus \tilde{C}^{p,q-1}$. Define $\tilde{\partial}': \tilde{C}^{p,q} \to \tilde{C}^{p-1,q}$ (resp. $\tilde{\partial}'': \tilde{C}^{p,q} \to \tilde{C}^{p,q-1}$) to be the projection of $\tilde{\partial}$ on the $\tilde{C}^{p-1,q}$ (resp. $\tilde{C}^{p,q-1}$) factor.

Of course, all the $\tilde{d}$, $\tilde{\partial}$, $\tilde{d}'$, $\tilde{d}''$, $\tilde{\partial}'$ and $\tilde{\partial}''$ are continuous and $\mathbf{r}$-module maps. Define d, ∂, d', d'', ∂' and ∂'' to be the restrictions of $\tilde{d}$, $\tilde{\partial}$, $\tilde{d}'$, $\tilde{d}''$, $\tilde{\partial}'$ and $\tilde{\partial}''$ (respectively) to the subspace $C(\mathbf{q},\mathbf{r}) = \sum_{p,q} C^{p,q}(\mathbf{q},\mathbf{r})$. We have the following relations:

$$(\mathrm{I}_{18}) \qquad (d')^2 = (d'')^2 = 0,$$

$$(\mathrm{I}_{19}) \qquad (\partial')^2 = (\partial'')^2 = 0.$$

The relation (I_{18}) (resp. (I_{19})) follows immediately from the corresponding fact for d (resp. ∂).

Now, we prove the following

3.1 Lemma. *With the notations as above,*

$$\tilde{\partial}'\tilde{d}'' + \tilde{d}''\tilde{\partial}' = \tilde{\partial}''\tilde{d}' + \tilde{d}'\tilde{\partial}'' = 0 \quad \text{on } \sum_{s \geqslant 0} \tilde{C}^s.$$

Proof. Let $\tilde{b}''$ (resp. $\tilde{b}'$) be the (degree $+1$) cochain map of $\tilde{C} = \sum_s \mathrm{Hom}(\Lambda^s(\mathbf{u} \oplus \mathbf{u}^-), \mathbf{C})$, considering $\mathbf{u} \oplus \mathbf{u}^-$ as a Lie algebra by demanding $[\mathbf{u},\mathbf{u}] = 0$ (resp. $[\mathbf{u}^-,\mathbf{u}^-] = 0$); $[\mathbf{u},\mathbf{u}^-] = 0$ and putting the subalgebra structure (coming from $\mathbf{q}$) on $\mathbf{u}^-$ (resp. $\mathbf{u}$). Define $\tilde{c}'' = \tilde{d}'' - \tilde{b}''$ (resp. $\tilde{c}' = \tilde{d}' - \tilde{b}'$). It is easy to see that $\tilde{d}''$, $\tilde{b}''$ and $\tilde{c}''$ are all derivations (of degree $+1$), are continuous and, further,

$$(\mathrm{I}_{20}) \qquad \tilde{c}''(\lambda) = (-1)^p \sum_{\varphi} (\mathrm{ad}\, y_\varphi)(\lambda) \otimes y_\varphi^* \quad \text{for } \lambda \in \mathrm{Hom}(\Lambda^p(\mathbf{u}), \mathbf{C}),$$

$$(\mathrm{I}_{21}) \qquad \tilde{c}''(\mu) = 0 \quad \text{for } \mu \in \mathrm{Hom}(\Lambda^q(\mathbf{u}^-), \mathbf{C}).$$

Here y_φ^* denotes the element of $\mathrm{Hom}(\mathbf{u}^-, \mathbf{C})$ defined by $y_\varphi^*(y_\phi) = \delta_{\varphi,\phi}$ for all $\varphi, \phi \in I$ ($\{y_\varphi\}_{\varphi \in I}$ is defined in §2) and $\mathbf{u}^- \oplus \mathbf{r}$ acts on

$$\mathrm{Hom}(\Lambda^p(\mathbf{q}/(\mathbf{u}^- \oplus \mathbf{r})), \mathbf{C}) \approx \mathrm{Hom}(\Lambda^p(\mathbf{u}), \mathbf{C})$$

under the adjoint action ($\mathbf{u}^- \oplus \mathbf{r}$ derives its Lie algebra structure from $\mathbf{q}$) (see, just before Lemma (2.4)). (For a fixed element $u \in \Lambda^p(\mathbf{u}) \otimes \Lambda^1(\mathbf{u}^-)$, $[\mathrm{ad}\, y_\varphi(\lambda) \otimes y_\varphi^*]u = 0$ for all but finitely many $\varphi \in I$, hence the expression $\sum \mathrm{ad}\, y_\varphi(\lambda) \otimes y_\varphi^*$ makes sense.) Since $\tilde{c}''$ is a derivation, we have by (I_{20}) and (I_{21}),

$$(\mathrm{I}_{22}) \qquad \tilde{c}''(\lambda \otimes \mu) = (-1)^p \sum_{\varphi \in I} (\mathrm{ad}\, y_\varphi)(\lambda) \otimes y_\varphi^* \wedge \mu$$

for $\lambda \in [\Lambda^p(\mathbf{u})]^*$ and $\mu \in [\Lambda^q(\mathbf{u}^-)]^*$. Further, the following identity is easy to see.

$$(\mathrm{I}_{23}) \qquad \tilde{\partial}'\tilde{b}'' + \tilde{b}''\tilde{\partial}' = 0 \quad \text{on } \tilde{C}^s.$$

Moreover, by (I_{22}) (for $Y \in \Lambda^p(\mathbf{u}^-)$ and $\mu \in \mathrm{Hom}(\Lambda^q(\mathbf{u}^-), \mathbf{C})$)

$$\begin{aligned}
\tilde{\partial}'\tilde{c}'' + \tilde{c}''\tilde{\partial}'(e(Y)\otimes\mu) &= (-1)^p\tilde{\partial}'\sum_\varphi \left[\mathrm{ad}\ y_\varphi(e(Y))\otimes y_\varphi^*\wedge\mu\right] \\
&\quad + \tilde{c}''\left[\tilde{\partial}'(e(Y))\otimes\mu\right] \\
&\qquad\qquad (\text{since } e(Y)\in \mathrm{Hom}(\Lambda^p(\mathbf{u}),\mathbf{C})) \\
&= (-1)^p\sum_\varphi\left(\tilde{\partial}'\left(\mathrm{ad}\ y_\varphi(e(Y))\right)\right)\otimes y_\varphi^*\wedge\mu \\
&\quad + (-1)^{p-1}\sum_\varphi\left(\mathrm{ad}\ y_\varphi(\tilde{\partial}'(e(Y)))\right)\otimes y_\varphi^*\wedge\mu \\
&\qquad\qquad (\text{by the continuity of } \tilde{\partial}') \\
&= (-1)^p\sum_\varphi e\left(\tilde{\partial}_0\left(\mathrm{ad}\ y_\varphi(Y)\right)\right)\otimes y_\varphi^*\wedge\mu \\
&\quad + (-1)^{p-1}\sum_\varphi e\left(\mathrm{ad}\ y_\varphi(\tilde{\partial}_0(Y))\right)\otimes y_\varphi^*\wedge\mu,
\end{aligned}$$

by using the invariance of e and the definition of $\tilde{\partial}'$. But $\tilde{\partial}_0(\mathrm{ad}\ y_\varphi(Y)) = \mathrm{ad}\ y_\varphi(\tilde{\partial}_0(Y))$ (as is easy to check), hence $\tilde{\partial}'\tilde{c}'' + \tilde{c}''\tilde{\partial}'(e(Y)\otimes\mu) = 0$. By continuity, we get

$$(\mathrm{I}_{24}) \qquad \tilde{\partial}'\tilde{c}'' + \tilde{c}''\tilde{\partial}' = 0 \quad \text{on } \tilde{C}^s.$$

Adding (I_{23}) and (I_{24}) we get $\tilde{\partial}'\tilde{d}'' + \tilde{d}''\tilde{\partial}' = 0$. Exactly similarly we can prove the other half. q.e.d.

Now, we can state one of the crucial propositions of this section.

3.2 Proposition. *The map* $H(\ker \partial', d'') \to H(C(\mathbf{q},\mathbf{r}), d'')$, *induced by the inclusion of* $\ker \partial'$ *in* $C(\mathbf{q},\mathbf{r})$ *is an isomorphism.*

Observe that $\ker \partial'$ is d'' stable, by the previous lemma.

Proof. For $\alpha = \sum n_i\alpha_i \in \Delta \cup \{0\}$, define $|\alpha| = |\alpha|_S = |\sum_{i\notin S} n_i|$. Further, define

$$\Lambda^s_{(k)}(\mathbf{u}^-) = \sum_{\substack{\text{all } \beta_j\in\Delta_+\setminus\Delta_+^S \\ \text{satisfying} \\ \sum_{j=1}^s|\beta_j|=k}} \mathbf{q}_{-\beta_1}\wedge\cdots\wedge\mathbf{q}_{-\beta_s} \subset \Lambda^s(\mathbf{u}^-)$$

(Δ_+^S is defined in §1.3). Put $\Lambda_{(k)}(\mathbf{u}^-) = \sum_{s\geq 0}\Lambda^s_{(k)}(\mathbf{u}^-)$. By [6, Proposition (5.4)], each $\Lambda^s_{(k)}(\mathbf{u}^-)$ is finite dimensional and, of course, $\Lambda^s_{(k)}(\mathbf{u}^-) = 0$ for $s > k$.

Denote

$$F_m = \sum_{p,q} \operatorname{Hom}_{\mathbf{r}}\Big(\Lambda^p(\mathbf{u}) \otimes \Lambda^q(\mathbf{u}^-)/\sum_{k \leqslant m-1} \Lambda^p(\mathbf{u}) \otimes \Lambda^q_{(k)}(\mathbf{u}^-), \mathbf{C}\Big) \hookrightarrow C(\mathbf{q}, \mathbf{r}).$$

(By definiton of $|\ |$, $\Lambda^s_{(k)}(\mathbf{u}^-)$ is an $\mathbf{r}$-module.)

We have the filtration

$$(\mathscr{F}_1) \qquad C(\mathbf{q}, \mathbf{r}) = F_0 \supset F_1 \supset \cdots \supset F_m \supset \cdots$$

since, for $\alpha, \beta, \alpha + \beta \in \Delta \cup \{0\}$, $|\alpha + \beta| \leqslant |\alpha| + |\beta|$, it is easy to see that all the F_m's are d'' closed.

Define $\hat{F}_m = \pi(F_m)$, where π is the canonical projection: $C(\mathbf{g}, \mathbf{r}) \to C(\mathbf{g}, \mathbf{r})/\ker \partial'$. This gives the filtration

$$(\mathscr{F}_2) \qquad C(\mathbf{g}, \mathbf{r})/\ker \partial' = \hat{F}_0 \supset \hat{F}_1 \supset \hat{F}_2 \supset \cdots.$$

Let $E_r^{m,n}$ denote the spectral sequence associated to the filtration $(\mathscr{F}_2)$. We want to compute $E_1^{m,n}$. By the definition, $E_1^{m,n} = H^{m+n}(\hat{F}_m/\hat{F}_{m+1})$.

Recall that for any $\mathbf{a}$-modules V and W ($\mathbf{a}$ is any Lie-algebra), the map ξ: $\operatorname{Hom}_{\mathbf{C}}(V, W^*) \to [V \otimes W]^*$, defined by $\xi(f)(v \otimes w) = f(v)w$ (for $f \in \operatorname{Hom}(V, W^*)$, $v \in V$ and $w \in W$), is an $\mathbf{a}$-module isomorphism.

Denote $Z^p = \ker \tilde{\partial}' \cap \tilde{C}^{p,0}$. $\hat{F}_m$ can be identified with

$$\sum_{p,q} \operatorname{Hom}_{\mathbf{r}}\Big(\Lambda^q(\mathbf{u}^-)\Big/\sum_{k \leqslant m-1} \Lambda^q_{(k)}(\mathbf{u}^-), \Lambda^p(\mathbf{u})^*/Z^p\Big).$$

This shows that $\hat{F}_m/\hat{F}_{m+1}$ can be identified (under the restriction map) with $\sum_{p,q} \operatorname{Hom}_{\mathbf{r}}(\Lambda^q_{(m)}(\mathbf{u}^-), \Lambda^p(\mathbf{u})^*/Z^p)$. The inclusion $\Lambda^q_{(m)}(\mathbf{u}^-) \hookrightarrow \Lambda^q(\mathbf{u}^-)$ has a canonical $\mathbf{r}$-module splitting. This allows us to identify $\sum_{p,q} \operatorname{Hom}_{\mathbf{r}}(\Lambda^q_{(m)}(\mathbf{u}^-), \Lambda^p(\mathbf{u})^*/Z^p)$ with a subspace of $\sum_{p,q} \operatorname{Hom}_{\mathbf{r}}(\Lambda^q(\mathbf{u}^-), \Lambda^p(\mathbf{u})^*/Z^p) \approx C(\mathbf{g}, \mathbf{r})/\ker \partial'$. Under this identification, $\sum_{p,q} \operatorname{Hom}_{\mathbf{r}}(\Lambda^q_{(m)}(\mathbf{u}^-), \Lambda^p(\mathbf{u})^*/Z^p)$ is b''-stable. (By (I_{23}), b'' descends to $C(\mathbf{g}, \mathbf{r})/\ker \partial'$.) From the expression (I_{22}), $c''(F_m) \subset F_{m+1}$, hence $c''(\hat{F}_m) \subset \hat{F}_{m+1}$. This gives

$$(\mathrm{I}_{25}) \qquad H_{d''}^{m+n}(\hat{F}_m/\hat{F}_{m+1}) \approx H_{b''}^{m+n}(\hat{F}_m/\hat{F}_{m+1}).$$

By the definition of b'' and (I_{25}), it is easy to see that

$$(\mathrm{I}_{26}) \qquad E_1^{m,n} \approx \sum_{p+q=m+n} \big[H^q_{(m)}(\mathbf{u}^-) \otimes \Lambda^p(\mathbf{u})^*/Z^p\big]^{\mathbf{r}},$$

where $H^*_{(m)}(\mathbf{u}^-)$ denotes the cohomology of the complex $\sum_q \operatorname{Hom}(\Lambda^q_{(m)}(\mathbf{u}^-), \mathbf{C})$.

Consider the embedding e_1: $\sum_p \Lambda^p(\mathbf{u}^-) \to \sum_p \Lambda^p(\mathbf{u})^*$, given by $(e_1(Y))X = \langle Y, X\rangle$ for $Y \in \Lambda^p(\mathbf{u}^-)$ and $X \in \Lambda^p(\mathbf{u})$. e_1 is an $\mathbf{r}$-module map and, of course, $e_1(\ker \partial_{\mathbf{u}^-}) \subset \sum_p Z^p(\partial_{\mathbf{u}^-}$ is the differential of the standard chain complex

$\Lambda(\mathbf{u}^-, \mathbf{C}))$. We have the following inclusion

$$(**)\qquad \left[H^q_{(m)}(\mathbf{u}^-) \otimes \left(\Lambda^p(\mathbf{u}^-)/(\ker \partial_{\mathbf{u}^-} \cap \Lambda^p(\mathbf{u}^-))\right)\right]^{\mathbf{r}} \xrightarrow{\mathrm{Id}\cdot \otimes e_1} \left[H^q_{(m)}(\mathbf{u}^-) \otimes \left(\Lambda^p(\mathbf{u})^*/Z^p\right)\right]^{\mathbf{r}}.$$

Using Corollary (2.3)(a), we see that $(**)$ is an isomorphism and the left side of $(**)$ is equal to 0 for all m, p and q. So, from (I_{26}), we get $E_1^{m,n} = 0$ for all m and n. Hence $E_\infty^{m,n} = 0$, as well. In particular, the canonical map $H(\hat{F}_m) \to H(C(\mathbf{g}, \mathbf{r})/\ker \partial')$ is surjective for all m. Using this we want to conclude that $H(C(\mathbf{g}, \mathbf{r})/\ker \partial', d'') = 0$. Fix $f \in C(\mathbf{g}, \mathbf{r})$ such that $d''f \in \ker \partial'$. We can choose $f_m \in F_m$ such that $\pi(f_m) - \pi(f) \in d''(C(\mathbf{g}, \mathbf{r})/\ker \partial')$. By the definition of topology on $C(\mathbf{g}, \mathbf{r})$, it is easy to see that the sequence $\{f_m\}_{m \in \mathbf{N}}$ converges to 0. Further, in the remark following Lemma (3.8), we prove that under the quotient topology on $C(\mathbf{g}, \mathbf{r})/\ker \partial'$, $d''(C(\mathbf{g}, \mathbf{r})/\ker \partial')$ is a closed subspace of $C(\mathbf{g}, \mathbf{r})/\ker \partial'$. So $\pi(f) \in d''(C(\mathbf{g}, \mathbf{r})/\ker \partial')$, implying that $H^*(C(\mathbf{g}, \mathbf{r})/\ker \partial', d'') = 0$. Of course, this immediately proves the proposition.

3.3 Remark. Define

$$G_m = \sum_{p,q} \mathrm{Hom}_{\mathbf{r}}\left(\Lambda^p(\mathbf{u}) \otimes \Lambda^q(\mathbf{u}^-) \Big/ \sum_{k_1 + k_2 \leqslant m-1} \Lambda^p_{(k_1)}(\mathbf{u}) \otimes \Lambda^q_{(k_2)}(\mathbf{u}^-), \mathbf{C}\right).$$

It is easy to see that G_m is d-closed and hence we get a filtration

$$(\kappa)\qquad C(\mathbf{q}, \mathbf{r}) = G_0 \supset G_1 \supset G_2 \supset \cdots.$$

Let $E_{\mathrm{r}}^{m,n}(\kappa)$ denote the corresponding spectral sequence. It is straightforward to see (the computation is exactly similar to the computation of $E_1^{m,n}(\mathscr{F}_2)$ in the proof of the previous proposition) that

$$E_1^{m,n}(\kappa) \approx \sum_{\substack{k_1, k_2 \\ k_1 + k_2 = m}} \sum_{\substack{p,q \\ p+q = m+n}} \left[H^p\left(\mathrm{Hom}\left(\Lambda_{(k_1)}(\mathbf{u}), \mathbf{C}\right)\right) \otimes H^q\left(\mathrm{Hom}\left(\Lambda_{(k_2)}(\mathbf{u}^-), \mathbf{C}\right)\right)\right]^{\mathbf{r}}.$$

Using Corollary (2.3)(a)and the fact that $H^p(\mathrm{Hom}(\Lambda_{(k_1)}(\mathbf{u}), \mathbf{C}))$ is $\mathbf{r}$-module isomorphic with $H_p(\Lambda_{(k_1)}(\mathbf{u}^-))$, we get $E_1^{m,n}(\kappa) = 0$ unless $m + n$ is even. In particular, the spectral sequence $E_{\mathrm{r}}^{m,n}(\kappa)$ degenerates at the E_1 term itself and hence $H^s(\mathbf{q}, \mathbf{r}, \mathbf{C})$ is isomorphic (although not canonically) with $\sum_{m+n=s} E_1^{m,n}(\kappa)$. So

$$(\mathrm{I}_{27})\qquad H^s(\mathbf{q}, \mathbf{r}, \mathbf{C}) \approx \begin{cases} 0 & \text{if } s \text{ is odd,} \\ [H^p(\mathbf{u}) \otimes H^p(\mathbf{u}^-)]^{\mathbf{r}} & \text{if } s = 2p. \end{cases}$$

Again using Corollary (2.3)(a), we get

$$(\mathrm{I}_{28})\qquad \begin{aligned}\dim_{\mathbf{C}} H^{2p}(\mathbf{q},\mathbf{r},\mathbf{C}) &= \text{number of irreducible } \mathbf{r}\text{-submodules in } H^p(\mathbf{u})\\ &= \text{number of elements of length } p \text{ in } W_S^1.\end{aligned}$$

This result is due to Lepowsky, who proved it by a different method (see [8, Corollary 6.7]). A sharper result is the content of our Theorem (3.15).

We derive the following

3.4 Lemma. *$d''\partial''u \in \ker \partial''$ for any $u \in C = C(\mathbf{q},\mathbf{r})$ implies $\partial''u = 0$. A similar statement is true with $''$ replace by $'$ throughout.*

Proof. Since $\Lambda^q(\mathbf{u}^-) = \Sigma_{k \geqslant 0} \Lambda^q_{(k)}(\mathbf{u}^-)$ (recall the notation $\Lambda_{(k)}(\mathbf{u}^-)$ from the proof of Proposition (3.2)), we have

$$\mathrm{Hom}(\Lambda^q(\mathbf{u}^-),\mathbf{C}) = \Pi_{k\geqslant 0}\mathrm{Hom}\big(\Lambda^q_{(k)}(\mathbf{u}^-),\mathbf{C}\big).$$

Denote, for any $p, q \geqslant 0$,

$$V_k^q = \prod_{m \leqslant k} \mathrm{Hom}(\Lambda^q_{(m)}(\mathbf{u}^-),\mathbf{C}); \qquad V_k^{q^\perp} = \prod_{m > k} \mathrm{Hom}(\Lambda^q_{(m)}(\mathbf{u}^-),\mathbf{C});$$
$$C_k^{p,q} = \mathrm{Hom}_{\mathbf{r}}(\Lambda^p(\mathbf{u}), V_k^q); \qquad C_k^{p,q^\perp} = \mathrm{Hom}_{\mathbf{r}}(\Lambda^p(\mathbf{u}), V_k^{q^\perp}).$$

We prove the lemma by contradiction. Suppose that $\partial''u \neq 0$. We can assume that u is homogeneous, i.e., $u \in C^{p,q+1}$ for some p, q. In the sequel, we would drop superscripts p, q if no confusion is likely. Let k_0 be the minimum integer such that $\partial''u \notin C_{k_0}^\perp$. Thus $\partial''u = v_1 + v_2$ with $v_1 \neq 0 \in \mathrm{Hom}_{\mathbf{r}}(\Lambda(\mathbf{u}), \Lambda_{(k_0)}(\mathbf{u}^-)^*)$ and $v_2 \in C_{k_0}^\perp$. This gives

$$(\mathrm{I}_{29})\qquad d''\partial''u = b''v_1 + c''v_1 + d''v_2,$$

where $b''v_1 \in \mathrm{Hom}_{\mathbf{r}}(\Lambda(\mathbf{u}), \Lambda_{(k_0)}(\mathbf{u}^-)^*)$ and, by (I_{22}), $c''v_1 + d''v_2 \in C_{k_0}^\perp$.

Further, we prove that $b''v_1 \neq 0$. We define a positive definite Hermitian form $\{\ ,\ \}$ on $\Lambda_{(k)}(\mathbf{u}^-)^*$ by transporting the Hermitian form $\{\ ,\ \}$ from $\Lambda_{(k)}(\mathbf{u})$ under the map $\bar{e}_1(k)$: $\Lambda_{(k)}(\mathbf{u}) \tilde{\to} \Lambda_{(k)}(\mathbf{u}^-)^*$, where $(\bar{e}_1(k)X)Y = \langle X, Y\rangle$ for $X \in \Lambda_{(k)}(\mathbf{u})$ and $Y \in \Lambda_{(k)}(\mathbf{u}^-)$. It is easy to see that $-\tilde{b}''$ is adjoint of $\tilde{\partial}''$ under $\{\ ,\ \}$, i.e.,

$$(\mathrm{I}_{30})\qquad \{\tilde{b}''f_1, f_2\} = -\{f_1, \tilde{\partial}''f_2\} \quad \text{for } f_1, f_2 \in \Lambda_{(k)}(\mathbf{u}^-)^*.$$

By choice, $u \in C^{p,q+1} = \prod_k \mathrm{Hom}_{\mathbf{r}}(\Lambda^p(\mathbf{u}), \Lambda^{q+1}_{(k)}(\mathbf{u}^-)^*)$. Define $u_{(k_0)}$ to be the projection of u on the k_0th factor. We have, for any $X \in \Lambda^p(\mathbf{u})$,

$$\begin{aligned}\left\{b''(v_1)(X), u_{(k_0)}(X)\right\} &= \pm\left\{v_1(X), \left(\partial'' u_{(k_0)}\right)X\right\} \quad (\text{by } (\mathrm{I}_{30}))\\ &= \pm\{v_1(X), v_1(X)\}\\ &\quad \left(\text{by using } \partial'' u = v_1 + v_2, \partial''\left(u_{(k_0)}\right) = (\partial'' u)_{(k_0)}\right)\\ &\neq 0 \quad \text{for some } X \in \Lambda^p(\mathbf{u})\\ &\qquad (\text{since, by assumption, } v_1 \neq 0).\end{aligned}$$

This establishes $b''v_1 \neq 0$. Moreover, by assumption, $d''\partial''u \in \mathrm{Ker}\,\partial''$, hence by (I_{29}), $\partial''d''\partial''u = 0 = \partial''b''v_1 + v$, with $v \in C^{\perp}_{k_0}$. But $\partial''b''v_1 \in \mathrm{Hom}_{\mathbf{r}}(\Lambda(\mathbf{u}), \Lambda_{(k_0)}(\mathbf{u}^-)^*)$, so $\partial''b''v_1 = 0$. Again using (I_{30}), we get

$$\{\partial''b''v_1(X), v_1(X)\} = \pm\{b''v_1(X), b''v_1(X)\} \neq 0$$

for some $X \in \Lambda^p(\mathbf{u})$, because $b''v_1 \neq 0$, a contradiction. This contradiction arose because we assumed $\partial''u \neq 0$. This proves the lemma.

Define the following operators on C:

$$\begin{aligned}S &= d\partial + \partial d, \qquad & S' &= d'\partial' + \partial' d', \qquad & S'' &= d''\partial'' + \partial'' d'';\\ L &= b\partial + \partial b, & L' &= b'\partial' + \partial' b', & L'' &= b''\partial'' + \partial'' b'',\end{aligned}$$

where b is defined as $b' + b''$.

By Lemma (3.1), we get

$$(\mathrm{I}_{31}) \qquad S = S' + S''.$$

Also, it is easy to prove

$$(\mathrm{I}_{32}) \qquad L = L' + L''.$$

3.5 Lemma. $S' = S'' = S/2$ *as operators on* $C = C(\mathbf{q}, \mathbf{r})$.

Proof. Recall the map $e\colon \sum_{s \geqslant 0} \Lambda^s(\mathbf{u} \oplus \mathbf{u}^-) \to \sum_{s \geqslant 0} [\Lambda^s(\mathbf{u} \oplus \mathbf{u}^-)]^*$ given by the nondegenerate form $\langle\ ,\ \rangle$, defined earlier. Also recall the identity (I_3), applied to $\Lambda(\mathbf{u})$, which states $\partial_{\mathbf{u}}(x_\varphi \wedge X) + x_\varphi \wedge \partial_{\mathbf{u}}(X) = -(\mathrm{ad}\, x_\varphi)X$ for any $X \in \Lambda(\mathbf{u})$ (x_φ, y_φ are defined in §2). Applying the map e, we get

$$(\mathrm{I}_{33}) \qquad \tilde{\partial}''\left(y_\varphi^* \wedge e(X)\right) + y_\varphi^* \wedge \tilde{\partial}''(e(X)) = -\left(\mathrm{ad}\, x_\varphi\right)e(X).$$

(For the notations y_φ^* and $\mathrm{ad}\, x_\varphi(\mu)$, see the proof of Lemma (3.1).)

Using the expressions (I_{22}) and (I_{33}), we get

$$(\mathrm{I}_{34}) \quad (\tilde{\partial}''\tilde{c}'' + \tilde{c}''\tilde{\partial}'')(e(Y) \otimes e(X)) = -\sum_{\varphi \in I} \mathrm{ad}\, y_\varphi(e(Y)) \otimes \mathrm{ad}\, x_\varphi(e(X)),$$

for $Y \in \Lambda^p(\mathbf{u}^-)$ and $X \in \Lambda^q(\mathbf{u})$.

Exactly similarly

$$(\tilde{\partial}'\tilde{c}' + \tilde{c}'\tilde{\partial}')(e(Y) \otimes e(X)) = -\sum_{\varphi \in I} \operatorname{ad} y_\varphi(e(Y)) \otimes \operatorname{ad} x_\varphi(e(X)).$$

So, by continuity, we have

$$(\mathrm{I}_{35}) \qquad \tilde{\partial}'\tilde{c}' + \tilde{c}'\tilde{\partial}' = \tilde{\partial}''\tilde{c}'' + \tilde{c}''\tilde{\partial}'' \quad \text{on } \tilde{C}.$$

Further, we show that $L' = L''$ on C. To prove this, we use our Theorem (2.1) in the special case when $L(\lambda_0)$ is the one-dimensional trivial module $\mathbf{C}$.

For $Y \in \Lambda^p(\mathbf{u}^-)$ and $X \in \Lambda^q(\mathbf{u})$, using the analogue of (I_{30}) for (b'', ∂'') replaced by (b', ∂'), we have

$$\begin{aligned}(\tilde{\partial}'\tilde{b}' + \tilde{b}'\tilde{\partial}')(e(Y) \otimes e(X)) &= (\tilde{\partial}'\tilde{b}' + \tilde{b}'\tilde{\partial}')(e(Y)) \otimes e(X) \\ &= -e(\Delta_{\mathbf{u}^-}(Y)) \otimes e(X),\end{aligned}$$

where $\Delta_{\mathbf{u}^-}$ is the Laplacian, operating on $\Lambda(\mathbf{u}^-)$, as defined in §2.

So, using Theorem (2.1), we get

$$(\mathrm{I}_{36}) \qquad \begin{aligned}&(\tilde{\partial}'\tilde{b}' + \tilde{b}'\tilde{\partial}')(e(Y) \otimes e(X)) \\ &\quad = -\tfrac{1}{2}[\sigma(\rho, \rho) - \sigma(\beta + \rho, \beta + \rho)]e(Y) \otimes e(X),\end{aligned}$$

for $Y \in W^\beta \subset \Lambda^p(\mathbf{u}^-)$, where W^β is an irreducible $\mathbf{r}$-module with highest weight β.

Exactly similarly

$$(\mathrm{I}_{37}) \qquad \begin{aligned}&(\tilde{\partial}''\tilde{b}'' + \tilde{b}''\tilde{\partial}'')(e(Y) \otimes e(X)) \\ &\quad = -\tfrac{1}{2}[\sigma(\rho, \rho) - \sigma(\beta' + \rho, \beta' + \rho)]e(Y) \otimes e(X),\end{aligned}$$

for $X \in \operatorname{Hom}(W^{\beta'}, \mathbf{C}) \subset \Lambda^q(\mathbf{u})$.

(I_{36}) and (I_{37}), put together, give (using continuity)

$$(\mathrm{I}_{38}) \qquad \partial'b' + b'\partial' = \partial''b'' + b''\partial'' \quad \text{on } C = [\tilde{C}]^{\mathbf{r}}.$$

Adding (I_{35}) and (I_{38}), the lemma follows.

3.6 Definition. Following Kostant [16, §2]; two operators $d, \partial\colon C \to C$ are called disjoint if the following holds.

(1) $d\partial X = 0$, for any $X \in C$, implies $\partial X = 0$ and

(2) $\partial dX = 0$, for any $X \in C$, implies $dX = 0$.

Now, we are in position to prove the following

3.7 Proposition. *$d', \partial'\colon C \to C$ are disjoint. Similarly $d'', \partial''\colon C \to C$ are disjoint.*

Proof. We would prove disjointness of d'', ∂'' (disjointness of d', ∂' is similar). By Lemma (3.4), it suffices to show that $\partial''d''u = 0$ implies $d''u = 0$.

Clearly $S''d''u = 0$. But $S' = S''$ (Lemma (3.5)), hence $(\partial'd' + d'\partial')d''u = 0$. Again using Lemma (3.4), this gives $\partial'(d''u) = 0$. By Proposition (3.2), $d''u = d''v$ for some $v \in \ker \partial'$. Further, c'' $(\ker \partial') \subset \text{Image}\, \partial'$. (This is easy to see, using the identity (I_{22}) and the following analogoue of (I_{33})).

$$\tilde{\partial}'\big(x_\varphi^* \wedge e(Y)\big) + x_\varphi^* \wedge \tilde{\partial}'(e(Y)) = -\text{ad}\, y_\varphi(e(Y)), \quad \text{for } Y \in \Lambda(\mathbf{u}^-).$$

Hence $d''v = b''v + c''v \in \text{Im}\, b'' + \text{Im}\, \partial'$. From the classical Hodge decomposition, since, by (I_{30}), $-b''$ is adjoint of ∂'' with respect to a positive definite Hermitian form, $\text{Im}\, L'' = \text{Im}\, b'' \oplus \text{Im}\, \partial''$. So $d''v \in \text{Im}\, b'' + \text{Im}\, \partial' \subset \text{Im}\, L'' + \text{Im}\, L' = \text{Im}\, L''$, by (I_{38}). But by assumption $\partial''d''v = 0$, so that $d''v \in \text{Im}\, L'' \cap \ker \partial'' = \text{Im}\, \partial''$, i.e., $d''u = d''v = \partial''\theta$ for some $\theta \in C$. Lemma (3.4) gives $\partial''\theta = 0$, proving the proposition.

3.8 Lemma. $\ker S \oplus \text{Im}\, S = C$.

Proof. By the disjointness of d' and ∂', $\ker S' \cap \text{Im}\, S' = (0)$. So, by Lemma (3.5), $\ker S \cap \text{Im}\, S = (0)$.

In the case C is finite dimensional, the lemma is immediate from dimensional considerations. For infinite dimensional C, we need to do more work.

By the classical Hodge decomposition, $L\ (= 2L')|_{\text{Im}\, L}: \text{Im}\, L \to \text{Im}\, L$ is an isomorphism. Let M denote its inverse. Define two operators $\tilde{M}$, R: $C \to C$ by

$$\tilde{M}|_{\text{Im}\, L} = M, \qquad \tilde{M}|_{\ker L} = 0, \qquad R = -\tilde{M}(S - L).$$

For any fixed $u \in C$ and $a \in \Lambda(\mathbf{u}) \otimes \Lambda(\mathbf{u}^-)$, there exists $j_0 = j_0(a)$ such that $(R^j u)a = 0$ for all $j \geqslant j_0$. To prove this, observe

(1) $\text{Hom}_{\mathfrak{r}}(\Lambda^\bullet(\mathbf{u}), \text{Hom}(\Lambda^\bullet_{(k)}(\mathbf{u}^-), \mathbf{C}))$ is stable under ∂'', b''.

(2) $c''(C_k^\perp) \subset C_{k+1}^\perp$ (use the identity (I_{22})). (See the proof of Lemma (3.4), for the notations.)

So, for any given k, $(R^{k+1}u) \in C_k^\perp$, hence $(R^j u)a = 0$ for all sufficiently large j. In particular, for any $u \in C$, $\Sigma_{j \geqslant 0}(R^j u)$ converges in C. (C is given the subspace topology from $\tilde{C}$.)

Now $\tilde{M}L|_{\text{Im}\, L} = \text{Id}|_{\text{Im}\, L}$, hence $\tilde{M}S|_{\text{Im}\, L} = \text{Id}|_{\text{Im}\, L} - R|_{\text{Im}\, L}$. Further, it can be seen that $\text{Im}\, L$ is closed in C. Hence $(\Sigma_{j \geqslant 0} R^j)(\text{Im}\, L) \subset \text{Im}\, L$. This gives that $\tilde{M}S|_{\text{Im}\, L} = \text{Id} - R|_{\text{Im}\, L}: \text{Im}\, L \to \text{Im}\, L$ is an (onto) isomorphism. So

$$(I_{39}) \qquad C = \text{Im}\, S + \text{Ker}\, L.$$

In particular (observing that $\ker L \subset \text{Ker}\, \partial$, from the Hodge decomposition for the operator L) $C = \text{Im}\, d + \ker \partial$, hence $\ker d \subset \text{Im}\, d + \ker S$. So

$$H(C, d) = \frac{\ker d}{\text{Im}\, d} \hookrightarrow \frac{\text{Im}\, d + \ker S}{\text{Im}\, d} \approx \frac{\ker S}{\ker S \cap \text{Im}\, d}.$$

Hence $\dim H^s(C, d) \leqslant \dim(\ker S|_{C^s})$. By Remark (3.3), $\dim H^s(C, d) = \dim H^s(C, \partial) = \dim \ker(L|_{C^s})$. But, by ($I_{39}$), $\dim(C^s/S(C^s)) \leqslant \dim \ker(L|_{C^s}) \leqslant \dim \ker(S|_{C^s})$.

Since $\ker S \cap \operatorname{Im} S = (0)$, the lemma follows.

Remark. We show that $d''(C(\mathbf{g}, \mathbf{r})/\ker \partial')$ is closed in $C(\mathbf{g}, \mathbf{r})/\ker \partial'$ ($C(\mathbf{g}, \mathbf{r})/\ker \partial'$ is equipped with the quotient topology).

Since L' and S' both commute with ∂', by Lemma (3.5) and identity (I_{38}), S and L both descend as operators on $C(\mathbf{g}, \mathbf{r})/\ker \partial'$. It is fairly easy to see that (from the Hodge decomposition with respect to ∂' and b') L: $C(\mathbf{g}, \mathbf{r})/\ker \partial' \to C(\mathbf{g}, \mathbf{r})/\ker \partial'$ is an isomorphism. Further, a proof similar to the one given above (of Lemma (3.8)) implies that S: $C(\mathbf{g}, \mathbf{r})/\ker \partial' \to C(\mathbf{g}, \mathbf{r})/\ker \partial'$ is again an isomorphism. In particular,

$$C(\mathbf{g}, \mathbf{r})/\ker \partial' = d''(C(\mathbf{g}, \mathbf{r})/\ker \partial') + \partial''(C(\mathbf{g}, \mathbf{r})/\ker \partial').$$

Let $f_n \in C(\mathbf{g}, \mathbf{r})$ be such that

$$\begin{aligned} \operatorname*{Lt.}_{n \to \infty} d''(\pi(f_n)) &= \pi(f), \text{ for some } f \in C(\mathbf{g}, \mathbf{r}) \\ &= d''(\pi(g)) + \partial''(\pi(h)), \text{ for some } g, h \in C(\mathbf{g}, \mathbf{r}). \end{aligned}$$

Taking d'', we get that $d''\partial''(h) \in \ker \partial'$, i.e., $\partial' d'' \partial''(h) = 0$. By Lemma (3.1), we get $d''\partial''(\partial' h) = 0$. Using Lemma (3.4), we have $\partial''\partial'(h) = 0$, i.e., $\partial'' h \in \ker \partial'$. So, $\text{Lt.}_{n \to \infty} d''(\pi(x_n)) = d''(\pi(g))$, proving the assertion.

3.9 Lemma (*Hodge type decomposition*). *Let C be any vector space* (*not necessarily finite dimensional*) *and d, ∂: $C \to C$ be two disjoint operators such that $d^2 = \partial^2 = 0$. Further, assume that*

$$(*) \qquad \ker S + \operatorname{Im} S = C \quad (\textit{where } S = d\partial + \partial d).$$

Then the following hold

(1) $\ker S = \ker d \cap \ker \partial$,

(2) $\ker S \cap \operatorname{Im} S = (0)$, i.e., $C = \ker S \oplus \operatorname{Im} S$,

(3) $\operatorname{Im} S = \operatorname{Im} d \oplus \operatorname{Im} \partial$ *and*

(4) *the canonical maps*

$$\ker S \to \ker d/\operatorname{Im} d \quad \textit{and} \quad \ker S \to \ker \partial/\operatorname{Im} \partial$$

are both isomorphisms.

Proof. Easy to verify.

3.10 Remark. (*) is automatically satisfied if C is finite dimensional and in this case this lemma is nothing but [16, Proposition (2.1)].

Now, we are in position to prove the following main theorem of this section.

3.11 Theorem. *Let* $A = (a_{ij})_{1 \leqslant i, j \leqslant l}$ *be a symmetrizable generalized Cartan matrix and let* S *be a subset of* $\{1, \cdots, l\}$, *of finite type. Let* $C = C(\mathbf{q}, \mathbf{r})$, *where* $\mathbf{q} = \mathfrak{g}(A)$ *is the Kac-Moody Lie algebra associated to* A, *and* $\mathbf{r} = \mathbf{r}_S$ *is the Lie subalgebra defined in* §(1.3). *Then* d *and* $\partial\colon C \to C$ *are disjoint.*

Of course, (d', ∂') (and (d'', ∂'')): $C \to C$ were shown to be disjoint in Proposition (3.7).

3.12 Remark. This theorem, when specialized to the case when $\mathbf{q}$ is a finite dimensional, semisimple Lie algebra, gives [17, Theorem 4.5].

Proof (of the theorem). $\partial u = \partial' u + \partial'' u$ (resp. $du = d'u + d''u$). By using Lemmas (3.5), (3.8) and (3.9) and Proposition (3.7), we get ∂u (resp. du) $\in \operatorname{Im} S$.

Now assume that $d\partial u = 0$. Then, clearly, $\partial u \in \ker S$. But ∂u is seen to be $\in \operatorname{Im} S$. So $\partial u = 0$.

Exactly similarly $\partial du = 0$ implies $du = 0$. q.e.d.

Lemmas (3.8) and (3.9) and Theorem (3.11) give, as an immediate corollary, the following.

3.13 Theorem. *Let* $C = C(\mathbf{q}, \mathbf{r})$ *be as in Theorem* (3.11). *Then, for the pair* (d, ∂), *the following hold*

(1) $\ker S = \operatorname{Ker} d \cap \ker \partial$,

(2) $C = \operatorname{im} d \oplus \operatorname{Im} \partial \oplus \ker S$ *and*

(3) *the canonical maps*

$$\psi_{d,S}\colon \ker S \to H(C, d) \quad \text{and} \quad \psi_{\partial,S}\colon \ker S \to H(C, \partial)$$

are both isomorphisms.

3.14 Remark. The above theorem is true with the pair (d, ∂) replaced by the pair (d', ∂') or the pair (d'', ∂'').

Let C be a bigraded vector space (i.e. $C = \sum_{p,q} C^{p,q}$) with an operator e: $C \to C$ satisfying $e^2 = 0$, of total degree ± 1. Define

$$H^{p,q}(C, e) = \left\{ a \in H(C, e) = \frac{\ker e}{\operatorname{Im} e} : (a + \operatorname{Im} e) \cap C^{p,q} \neq \varnothing \right\}.$$

$H(C, e)$ is said to be *bigraded* if

$$H(C, e) = \bigoplus_{p,q} H^{p,q}(C, e).$$

Now, let us consider the situation when $C = C(\mathbf{q}, \mathbf{r})$ and e is any one of the differentials d, ∂, d', d'', ∂' or ∂''. $C(\mathbf{q}, \mathbf{r})$ is bigraded by $C(\mathbf{q}, \mathbf{r}) = \sum_{p,q} C^{p,q} = \sum_{p,q} \operatorname{Hom}_{\mathbf{r}}(\Lambda^p(\mathbf{u}) \otimes \Lambda^q(\mathbf{u}^-), \mathbf{C})$. Also, the operators d', d'', ∂' and ∂'' are of pure bidegrees $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$ respectively. Hence S', S'' are operators of bidegree $(0, 0)$. But more important is the fact that S (by Lemma (3.5)) is of bidegree $(0, 0)$. In particular, in view of Theorem (3.13), this implies that $H(C, d)$ (resp. $H(C, \partial)$) is bigraded and $\psi_{d,S}$ (resp. $\psi_{\partial,S}$) is an isomorphism of bidegree $(0,0)$. Further $H(C, \partial)$ is relatively easy to describe. This gives the following generalization of [17, Corollary 5.3.4], to all Kac-Moody Lie algebras and any parabolics of finite type.

3.15 Theorem. *Let* $C = C(\mathbf{q}, \mathbf{r})$ *be as in Theorem* (3.11). *Then*

(1) $H(C, d)$ *is bigraded,*

(2) $H^{p,q}(C, d) \approx H^{p,q}(C, d'')$ *and*

(3) $H^{p,q}(C, d) = 0$ *for* $p \neq q$ *and, for any* $p \geqslant 0$, $\dim_{\mathbf{C}} H^{p,p}(C, d) =$ *the number of elements of length* p *in* W_S^1 (*see* §1.4 *for the notation*) = *number of* $\mathbf{r}$-*irreducible components in* $H_p(\mathbf{u}^-, \mathbf{C})$.

Proof. (2) is easy since $S = 2S''$. Now, $H^{p,q}(C, d) \approx \ker S \cap C^{p,q} \approx H^{p,q}(C, \partial)$. Further, by the definition of ∂ and finite dimensionality of $H_p(\mathbf{u}^-, \mathbf{C})$,

$$\begin{aligned} H^{p,q}(C, \partial) &\approx \left[H_q(\mathbf{u}, \mathbf{C}) \otimes H_p(\mathbf{u}^-, \mathbf{C})\right]^{\mathbf{r}} \quad \text{(notice the shift in } (p, q)) \\ &\approx \left[\operatorname{Hom}\left(H_q(\mathbf{u}^-, \mathbf{C}), \mathbf{C}\right) \otimes H_p(\mathbf{u}^-, \mathbf{C})\right]^{\mathbf{r}} \\ &\approx \operatorname{Hom}_{\mathbf{r}}\left(H_q(\mathbf{u}^-, \mathbf{C}), H_p(\mathbf{u}^-, \mathbf{C})\right). \end{aligned}$$

Now the theorem follows using Corollary (2.3)(a).

3.16 Remark. Lemma (3.5) and properties (1) and (2) in the above theorem are the analogues of the corresponding properties for compact Kahler manifolds (see, e.g., [8, Chapter 0, §7]).

Finally, we have the following proposition, which determines $\operatorname{Ker} S$ more explicitly, although we would not be using it in this paper.

3.17 Proposition. *Let* $h \in \operatorname{Ker} L$. *Then of course* h *determines an element* $[h]$ *in* $H(C, \partial)$. *Let* $s(h) = \psi_{\partial,S}^{-1}([h])$ (*see Theorem* (3.13) *for the notation*). *Then* $s(h) = \sum_{j \geqslant 0} R^j(h)$. ($R$ *is defined in the proof of Lemma* (3.8).)

Proof. There exists a $u \in \operatorname{Im} L$ such that $s(h) = h + u$. It suffices to show that $(1 - R)s(h) = h$:

$$\begin{aligned} (1 - R)s(h) &= (1 + \tilde{M}S - \tilde{M}L)s(h) \\ &= (1 - \tilde{M}L)s(h) \quad (\text{since } s(h) \in \operatorname{Ker} S) \\ &= h \quad \left(\text{since } h \in \operatorname{Ker} L \text{ and } \tilde{M}L|_{\operatorname{Im} L} = \operatorname{Id}|_{\operatorname{Im} L}\right). \end{aligned}$$

4. Identification of the dual of Schubert basis for G/P with a d, ∂-harmonic basis

Let $\mathbf{q} = \mathbf{q}(A)$ be the Kac-Moody Lie algebra associated to a symmetrizable generalized Cartan matrix $A = (a_{ij})_{1 \leqslant i, j \leqslant l}$ and let S be a subset of $\{1, \cdots, l\}$, of finite type. Let G denote the (possibly infinite dimensional) affine algebraic group/$\mathbf{C}$ in the sense of Šafarevič, associated to the first derived Lie algebra $\mathbf{q}^1$ (see §1.9 for details). Recall, from §1.5, the conjugate linear involution ω_0 of $\mathbf{q}$. On "integration", this gives rise to an involution of G. Let K denote the fixed point set of this involution.

The subgroup of $\mathrm{Aut}_{\mathbf{C}}(\mathbf{h})$ generated by the reflections $\{\bar{r}_i\}_{1 \leqslant i \leqslant l}$ (respectively $\{\bar{r}_i\}_{i \in S}$) is denoted by $\overline{W}$ (resp. $\overline{W}_S$), where $\bar{r}_i(h) = h - \alpha_i(h)h_i$, for all $h \in \mathbf{h}$. It is easy to see that, under the canonical identification χ: $\mathrm{Aut}\,\mathbf{h} \xrightarrow{\sim} \mathrm{Aut}(\mathbf{h}^*)$ (given by $(\chi f)\varphi(h) = \varphi(f^{-1}h)$, for $f \in \mathrm{Aut}\,\mathbf{h}$, $\varphi \in \mathbf{h}^*$ and $h \in \mathbf{h}$), $\overline{W}$ corresponds with W, in fact $\chi(\bar{r}_i) = r_i$ for all $1 \leqslant i \leqslant l$. From now on, we would identify $\overline{W}$ with W (under χ) and use the same symbol W for both.

For each $1 \leqslant i \leqslant l$, there exists a unique homomorphism ϕ_i: $\mathrm{SL}_2(\mathbf{C}) \to G$, satisfying $\phi_i\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) = \exp(te_i)$ and $\phi_i\left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right) = \exp(tf_i)$ (for all $t \in \mathbf{C}$) (see §1.1 and 1.9 for various notations). Define

$$H_i^+ = \phi_i\left\{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t > 0\right\}, \quad H_i = \phi_i\left\{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in \mathbf{C}^*\right\},$$

$$G_i = \phi_i(\mathrm{SL}_2(\mathbf{C})), \qquad N_i = \text{Normalizer of } H_i \text{ in } G_i,$$

$$H^+ \text{ (resp. } H) = \text{the subgroup (of } G) \text{ generated by } H_i^+ \text{ (resp. all } H_i),$$

$$N = \text{the subgroup (of } G) \text{ generated by all } N_i.$$

There is an isomorphism φ: $W \to N/H$, such that $\varphi(r_i)$ is the coset $N_iH \setminus H$, mod H (see [13, §2]). In the sequel, we would identify W with N/H under φ.

Put $B = HU$, $B^- = HU^-$ (U, U^- are defined in §1.9) and $P = P_S = BW_SB$ (P_S are called the standard parabolics, corresponding to the subset $S \subset \{1, \cdots, l\}$, containing B). Denote by K_S the subgroup $K \cap P_S$. It is easy to see, using [14, Theorem 4(d)], that the canonical inclusion $K/K_S \to G/P_S$ is a (surjective) homeomorphism. ($K \subset G$ is given the subspace topology and topology on G is described in §1.9.)

4.1 Bruhat decomposition [13], [14]. Recall the definition of W_S^1 from §1.4. W_S^1 can be characterized as the set of elements of minimal length in the cosets W_Sw ($w \in W$), (each such coset contains a unique element of minimal length).

G can be written as disjoint union

$$G = \bigcup_{w \in W_S^1} \left(Ua(w)^{-1}P_S\right)$$

($a(w)$ is an element of N satisfying $a(w) \bmod H = \varphi(w)$; in fact, we will choose $a(w) \in N \cap K$, which is possible because $KH \supset N$), so that

$$G/P_S = \bigcup_{w \in W_S^1} \left(Ua(w)^{-1}P_S/P_S\right).$$

As announced in [14, Theorem 4], G/P_S is a CW complex with cells $\{V_w = Ua(w)^{-1}P_S/P_S\}_{w \in W_S^1}$ and $\dim_{\mathbf{R}} V_w = 2\,\text{length}\, w$. (To interchange right and left cosets we have, in the expression of V_w, $a(w)^{-1}$ instead of $a(w)$ as in [14].) We describe V_w more explicitly.

Define, for any $w \in W$,

$$U_w = \left(a(w)Ua(w)^{-1}\right) \cap U^- \quad \text{and} \quad \tilde{U}_w = \left(a(w)Ua(w)^{-1}\right) \cap U.$$

Using [13, Corollary 5], it is easy to see that $a(w)Ua(w)^{-1} = U_w \cdot \tilde{U}_w$, so that

$$Ua(w)^{-1}P_S = a(w)^{-1}a(w)Ua(w)^{-1}P_S = a(w)^{-1}U_wP_S\,(\text{since } \tilde{U}_w \subset U \subset P_S).$$

Now for $w \in W_S^1$:

(1) U_w is a finite dimensional, connected, simply connected, unipotent, complex group (with the subspace topology) of complex dim. = length w and

(2) $U_w \cap P_S = (e)$. Hence the canonical map

$$i^w \colon a(w)^{-1}U_w \to a(w)^{-1}U_wP_S/P_S = V_w$$

is a homeomorphism

4.2 A canonical basis for Ker L. Recall the definition of the operator L: $C = C(\mathbf{q}, \mathbf{r}) \to C$ from §3 (just before Lemma (3.5)). We describe a basis for Ker L.

Fix a $w \in W_S^1$ of length p. Define $\Phi_w = w\Delta_- \cap \Delta_+$. Φ_w consists of only real roots (say $\{\beta_1, \cdots, \beta_p\}$) (see [6, Proposition 2.2]). Pick $y_{\beta_i} \in \mathbf{q}_{-\beta_i}$ (β_i being real, $\mathbf{q}_{-\beta_i}$ is one dimensional) of unit norm with respect to the form $\{\ ,\ \}$ and let $x_{\beta_i} = -\omega_0(y_{\beta_i})$. Let $M(w\rho - \rho) \subset \Lambda^p(\mathbf{u}^-)$ be *the* (see §2.3(a)) irreducible $\mathbf{r}$-submodule with highest weight $(w\rho - \rho)$. The corresponding highest weight vector can be easily seen, using [6, Proposition (2.5)], to be $y_{\beta_1} \wedge \cdots \wedge y_{\beta_p}$. There exists a unique element $\bar{h}^w \in [M(w\rho - \rho) \otimes \Lambda^p(\mathbf{u})]^{\mathbf{r}}$, such that $\bar{h}^w = (2i)^p(y_{\beta_1} \wedge \cdots \wedge y_{\beta_p} \wedge x_{\beta_1} \wedge \cdots \wedge x_{\beta_p}) \bmod P_w \otimes \Lambda^p(\mathbf{u})$, where P_w is the orthogonal (under $\{\ ,\ \}$) complement of $y_{\beta_1} \wedge \cdots \wedge y_{\beta_p}$ in $M(w\rho - \rho)$ ($\subset \Lambda^p(\mathbf{u}^-)$). Denote $\bar{r}^w = (2i)^p(y_{\beta_1} \wedge \cdots \wedge y_{\beta_p} \wedge x_{\beta_1} \wedge \cdots \wedge x_{\beta_p})$ and denote $e(\bar{r}^w)$ (resp. $e(\bar{h}^w)$) by r^w (resp. h^w). (e is defined in the beginning of §3.) It is easy to see that $h^w \in \ker L$. Moreover $\{h^w\}_{w \in W_S^1 \text{ with length } w = p}$ is a $\mathbf{C}$-basis of $\ker L|_{C^{2p}(\mathbf{q}, \mathbf{r})}$.

Finally define (see Proposition (3.17)) $s^w = \psi_{\partial,S}^{-1}([h^w])$, where $[h^w]$ denotes the element in $H(C, \partial)$, determined by h^w and $\psi_{\partial,S}$ is defined in Theorem (3.13).

4.3 Definition. Let M be a finite-dimensional smooth ($= C^\infty$) manifold. A map $f\colon M \to K/K_S$ is said to be *smooth* if, for all $x_0 \in M$, there exists an open neighborhood $N_{x_0} \subset M$ and a continuous map $\tilde{f}\colon N_{x_0} \to K$ satisfying

(1) $\pi \circ \tilde{f} = f|_{N_{x_0}}$, where $\pi\colon K \to K/K_S$ is the canonical projection and

(2) consider the composite of the maps

$$N_{x_0} \xrightarrow{\tilde{f}} K \hookrightarrow G \xrightarrow{i} \mathfrak{A}$$

(i is defined in §1.9). Since $i \circ \tilde{f}\colon N_{x_0} \to \mathfrak{A}$ is continuous, may be after shrinking N_{x_0}, there exists a finite-dimensional vector subspace $F \subset \mathfrak{A}$ such that $i \circ \tilde{f}(N_{x_0}) \subset F$.

We demand that $i \circ \tilde{f}\colon N_{x_0} \to F$ is a smooth map in the usual sense. Such a lift $\tilde{f}\colon N_{x_0} \to K$ will be called a *local smooth lift*.

4.4 Definition. Let M be a finite-dimensional smooth manifold. Given a $u \in C^s(\mathfrak{q}, \mathfrak{r})$ and a smooth map $f\colon M \to K/K_S$, we construct a smooth s-form $f^*(u)$ on M as follows.

Fix a $x_0 \in M$. Choose a local smooth lift $\tilde{f}\colon N_{x_0} \to K$. Consider the map $i \circ L_{\tilde{f}(x_0)^{-1}} \circ \tilde{f}\colon N_{x_0} \to \mathfrak{A}$, where $L_{\tilde{f}(x_0)^{-1}}$ is the left translation (by $\tilde{f}(x_0)^{-1}$): $K \to K$. Define $(f^*u)_{x_0} = (i \circ L_{\tilde{f}(x_0)^{-1}} \circ \tilde{f})^*\tilde{u}$, where $\tilde{u}$ is any translation invariant s-form on $\mathfrak{A}$ (so that $\tilde{u}$ is given by $\tilde{u}_0 \in \operatorname{Hom}_{\mathbf{C}}(\Lambda^s(\mathfrak{A}), \mathbf{C})$) satisfying $\tilde{u}_0|_{\Lambda^s(\mathfrak{q}^1)} = u|_{\Lambda^s(\mathfrak{q}^1)}$; here, we are using the imbedding of $\mathfrak{q}^1$ in $\mathfrak{A}$ via $\dot{i}$ (see §1.9).

It is a routine checking that $f^*(\mathbf{u})$ is well defined, i.e., $(f^*\mathbf{u})_{x_0}$ does not depend upon the particular choices of $\tilde{f}$; $\tilde{u}$ and further $f^*\mathbf{u}$ is a smooth s-form on M.

Now, we can state the main theorem of this section, which is a generalization of [17, Theorem (6.15)].

4.5 Theorem. *Let $\mathfrak{q} = \mathfrak{q}(A)$ be the Kac-Moody Lie algebra associated to a symmetrizable generalized Cartan matrix $A = (a_{ij})_{1 \leqslant i, j \leqslant l}$ and let $S \subset \{1, \cdots, l\}$ be a subset of finite type. Recall the corresponding group G (defined in* §1.9) *and $P = P_S$, defined earlier in this section.*

For any w (of length p) $\in W_S^1$, we have the Bruhat cell (of real dimension $2p$)

$$a(w)^{-1}U_w \xrightarrow[\sim]{i^w} Ua(w)^{-1}P_S/P_S = V_w$$

in G/P_S (defined in §4.1) *and also, we have defined a d, ∂ harmonic form $s^w \in C^{2p}(\mathfrak{q}, \mathfrak{r})$ in* §4.2.

Then $i^w\colon a(w)^{-1}U_w \to V_w \hookrightarrow G/P_S \cong K/K_S$ is a smooth map and $(i^w)^*(s^{w'})$ is an integrable $2p$-form (see §4.3 *and* 4.4 *for definitions), for any w, $w' \in W_S^1$ of length p. Further $\int (i^w)^* s^{w'} = 0$ if $w \neq w' \in W_S^1$ and $\int (i^w)^* s^w > 0$ for all $w \in W_S^1$ (U_w is oriented by its complex structure and sign convention is that of*

[15, *Chapter* IX, *Proposition* (2.1)]). *In fact the form* $(i^w)^*s^{w'}$ *itself is identically zero, if* $w \neq w' \in W_S^1$ *with* $l(w) = l(w')$ *and*

$$\int (i^w)^*s^w = 2^{(2p)}\int_{U_w} (\exp 2(w\rho - \rho))h(g)\,dg$$

(*the map* $h(g)$ *is defined in the proof*).

Proof. We first show that i^w is a smooth map. The map $K \times H^+ \times U \to G$ (defined by $(k, h, u) \mapsto khu$) is a homeomorphism [14, Theorem 4]. So, any $g \in G$ can be uniquely expressed as $g = k(g)h(g)u(g)$ with $k(g) \in K$, $h(g) \in H^+$ and $u(g) \in U$. Let k: $G \to K$ be the projection on the first factor. We have a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{i} & \prod_{i=1}^{l}(L(\Lambda_i) - (0)) \times \prod_{i=1}^{l}(L^*(\Lambda_i) - (0)) \\ k\downarrow & & \downarrow\hat{k} \\ K & \xrightarrow{i_{|K}} & \prod_{i=1}^{l}(L(\Lambda_i) - (0)) \times \prod_{i=1}^{l}(L^*(\Lambda_i) - (0)) \end{array}$$

where $\hat{k}$ is the map defined by

$$\sum_{i=1}^{l} v_i + \sum_{i=1}^{l} w_i \mapsto \sum_{i=1}^{l} v_i/\|v_i\|_i + \sum_{i=1}^{l} \bar{v}_i/\|v_i\|_i$$

for $v_i \in L(\Lambda_i) - (0)$, and $w_i \in L^*(\Lambda_i) - (0)$. (Here $\|\ \|_i$ comes from a K-invariant, positive definite, normalized (so that $\|v_{\Lambda_i}\|_i = 1$) Hermitian form on $L(\Lambda_i)$ and $\bar{v}_i$ denotes the conjugate of v_i under the canonical conjugation on $L(\Lambda_i)$, obtained by its real form $U(\mathbf{q}(R))v_{\Lambda_i}$. Further $\bar{v}_i/\|v_i\|$ is to be considered as an element in $L^*(\Lambda_i) = L(\Lambda_i)$.)

This proves that i^w is a smooth map. Now, we compute $\int_{a(w)^{-1}U_w} i^{w*}(s^{w'})$. Since the map i^w: $a(w)^{-1}U_w \to G/P$ factors through G/B and $s^{w'} \in C^{2p}(\mathbf{q}, \mathbf{r})$ restricts to the d, ∂ harmonic form (corresponding to w') in $C^{2p}(\mathbf{q}, \mathbf{h})$, it suffices to compute this integral for the case $S = \varnothing$, i.e., $P_S = B$.

Define $\bar{i}^w$: $U_w \to G/B = K/T\,(T = B \cap K)$ by $\bar{i}^w(g) = gB$. Since $a(w) \in K$ and $s^{w'}$ is a K-invariant form, we have

$$\int_{a(w)^{-1}U_w} (i^w)^*s^{w'} = \int_{U_w} (\bar{i}^w)^*s^{w'}.$$

The Lie algebra $\mathbf{u}_w$ (of U_w) can be easily seen to be $\sum_{\alpha \in -\Phi_w = w\Delta_+ \cap \Delta_-} \mathbf{q}_\alpha$. Let Vol denote the volume form with respect to the left invariant metric on U_w, induced by the restriction of the Hermitian form $\{\ ,\ \}$ (on $\mathbf{q}$) to $\mathbf{u}_w$.

There is associated the adjoint representation Ad: $G \to \operatorname{Aut} \mathbf{q}$ (see, for the details, [13, §2]). $\operatorname{Ad}|_B$ keeps $\mathbf{b}$-stable. Let $\widetilde{\operatorname{Ad}}: B \to \operatorname{Aut}(\mathbf{q}/\mathbf{b}) = \operatorname{Aut}(\mathbf{n}^-)$ be the map, induced by the restriction of Ad (to B). We extend (again denoted by) $\widetilde{\operatorname{Ad}}: B \to \operatorname{Aut}_{\mathbf{C}}(\mathbf{n} \oplus \mathbf{n}^-)$, by

$$\widetilde{\operatorname{Ad}}(b)(v_1 + v_2) = w_0 \widetilde{\operatorname{Ad}}(b)(w_0 v_1) + \widetilde{\operatorname{Ad}}(b)(v_2)$$
$$\text{for } b \in B,\ v_1 \in \mathbf{n} \text{ and } v_2 \in \mathbf{n}^-.$$

This gives rise to the dual representation (again denoted by) $\widetilde{\operatorname{Ad}}$: $B \to \operatorname{Aut}_{\mathbf{C}}(\operatorname{Hom}_{\mathbf{C}}(\mathbf{n} \oplus \mathbf{n}^-, \mathbf{C}))$. It is easy to see that for any $u \in U$ and $h \in H$

(1) $(\widetilde{\operatorname{Ad}}u)\theta - \theta \in \operatorname{Im} \tilde{\partial}$, for $\theta \in \ker \tilde{\partial}$, and

(2) $(\widetilde{\operatorname{Ad}}h)(\operatorname{Im} \tilde{\partial}) \subset \operatorname{Im} \tilde{\partial}$

($\tilde{\partial}$ is defined in the beginning of §3). (To prove (1), use (I_3) together with the definition of U as the group generated by U_α's, $\alpha \in \Delta_+^{re}$. See §1.9.)

Now, fix a $g \in U_w$. The integrand

$$(\mathrm{I}_{40}) \qquad [(\bar{i}^w)^* s^{w'}]_g = \{[L_g^*(\bar{i}^w)^* s^{w'}]_e, \operatorname{Vol}_e\} \operatorname{Vol}_g.$$

By the "left K-invariance" of $s^{w'}$, we have

$$(\mathrm{I}_{41}) \qquad [(\bar{i}^w)^* s^{w'}]_g = \{[\bar{L}_{b(g)}^*(s^{w'})]_e, \operatorname{Vol}_e\} \operatorname{Vol}_g,$$

where $\bar{L}_{b(g)}$: $U_w \to G/B$ is the map $\bar{L}_{b(g)}(a) = b(g)aB = b(g)ab(g)^{-1}B$ for all $a \in U_w$.

It is easy to see that

$$(\mathrm{I}_{42}) \qquad \{[\bar{L}_{b(g)}^*(s^{w'})]_e, \operatorname{Vol}_e\} = s^{w'}(\widetilde{\operatorname{Ad}} b(g) \bar{r}^w) = (\widetilde{\operatorname{Ad}}(b(g)^{-1}) s^{w'}) \bar{r}^w,$$

where $\bar{r}^w$ is defined in §4.2.

Further, for any $f \in \operatorname{Hom}(\Lambda^s(\mathbf{n} \oplus \mathbf{n}^-), \mathbf{C})$ and $v = v_1 \wedge \cdots \wedge v_s \in \Lambda^s(\mathbf{n} \oplus \mathbf{n}^-)$

$$(\mathrm{I}_{43}) \qquad f(v) = \{f, e(-w_0 v_1) \wedge \cdots \wedge e(-w_0 v_s)\},$$

($\{\ ,\ \}$ on $\operatorname{Hom}(\Lambda(\mathbf{n} \oplus \mathbf{n}^-), \mathbf{C})$ is defined by transporting $\{\ ,\ \}$ from $\Lambda(\mathbf{n} \oplus \mathbf{n}^-)$ via e; see also the proof of Lemma (3.4)).

By (I_{43}) we get

$$(\mathrm{I}_{44}) \qquad (\widetilde{\operatorname{Ad}}(b(g)^{-1}) s^{w'}) \bar{r}^w = \{\widetilde{\operatorname{Ad}}(b(g)^{-1})(s^{w'}), r^w\}.$$

But, $s^{w'} = r^{w'} + \tilde{\partial}\theta$ for some $\theta \in C(\mathbf{q}, \mathbf{r})$, since $r^{w'} = h^{w'}$ (by assumption $S = \varnothing$). Hence

$$\widetilde{\operatorname{Ad}}(u(g)^{-1}) \widetilde{\operatorname{Ad}}(h(g)^{-1})(r^{w'} + \tilde{\partial}\theta) = \widetilde{\operatorname{Ad}}(h(g)^{-1}) r^{w'} + \tilde{\partial}\theta',$$

for some $\theta' \in \operatorname{Hom}_{\mathbf{C}}(\Lambda^{2p+1}(\mathbf{n} \oplus \mathbf{n}^-), \mathbf{C})$. By ($I_{42}$) and ($I_{44}$), we get

$$\left\{\left[\bar{L}^*_{b(g)}(s^{w'})\right]_e, \mathrm{Vol}_e\right\} = \left\{\widetilde{\mathrm{Ad}}\left(h(g)^{-1}\right)r^{w'} + \tilde{\partial}\theta', r^w\right\}$$
$$= \left\{\widetilde{\mathrm{Ad}}\left(h(g)^{-1}\right)r^{w'}, r^w\right\},$$

since $\tilde{\partial}^*(r^w) = \tilde{b}(r^w) = 0$. Finally

$$\left\{\widetilde{\mathrm{Ad}}\left(h(g)^{-1}\right)r^{w'}, r^w\right\} = \begin{cases} 0 & \text{if } w \neq w', \\ 2^{2p}(\exp 2(w\rho - \rho))h(g) & \text{if } w = w'. \end{cases}$$

This proves the theorem.

Added in proof.

(1) As an application of the results in the paper, we prove that G/P_S is a formal space in the sense of rational homotopy theory. Further, we have explicitly determined the minimal model of G/B and the structure of $\pi_*(G/B) \otimes_{\mathbf{Z}} \mathbf{C}$ as a Lie algebra under Whitehead product.

(2) We also prove that $H^*(\mathbf{q}, \mathbf{r}_S)$ (resp. $H^*(\mathbf{q}^1)$) is canonically isomorphic (as a graded algebra) with the singular cohomology algebra $H^*(G/P_S, \mathbf{C})$ (resp. $H^*(G, \mathbf{C})$). Moreover, the isomorphism is explicitly given by an 'integration' map.

(3) The author, together with Kostant, has determined the value of the integral in Theorem 4.5 and show that it is equal to

$$(-4\pi)^p \prod_{\phi \in \Phi_w = w\Delta_- \cap \Delta_+} \sigma(w\rho, \phi)^{-1}.$$

We, together, also prove that there is a filtration of the ring $\mathbf{C}[W_S^1]$ ($\mathbf{C}[W_S^1]$ is the space of *all* the functions: $W_S^1 \to \mathbf{C}$. This is made into a ring under pointwise multiplication), such that the corresponding grade algebra $Gr\mathbf{C}[W_S^1]$ is naturally isomorphic (as an algebra) with the singular cohomology $H^*(G/P_S, \mathbf{C})$. Exactly similar result is true with G/P_S replaced by any left B invariant closed subvariety $V_{\mathscr{H}}$ of G/P_S. In this case, W_S^1 gets replaced by a subset $\mathscr{H}$ of W_S^1, defined by $\mathscr{H} = \{w \in W_S^1 \colon w \bmod P_S \in V_{\mathscr{H}}\}$. In particular, this result holds for closures of Bruhat cells $BwP_S|P_S$.

In the case $S = \varnothing$, so that $P_S = B$, the isomorphism of $Gr\mathbf{C}[W]$ with $H^*(G/B, \mathbf{C})$ is also shown to be W-equivariant, under the action of W on $\mathbf{C}[W]$ by $(w \cdot f)\eta = f(w^{-1}\eta)$, for $w, \eta \in W$ and $f \in \mathbf{C}[W]$. Of course, the action of $W \approx N(T)/T$ on $H^*(G/B, \mathbf{C})$ is induced from the action of $N(T)$ on the space $G/B \approx K/T$ defined as follows.

$$n \cdot (k \bmod T) = (nkn^{-1}) \bmod T, \quad \text{for } n \in N(T) \text{ and } k \in K.$$

We use these results to determine the cup product of any two cohomology classes in $H^*(G/B)$.

(4) The author learned that Kac-Peterson have also proved the Proposition 2.10 in their recent paper: *Unitary structure in representations of infinite-dimensional groups and a convexity theorem*, Invent. Math. **76** (1984) 1–14.

(5) The author discovered that he has used the same symbol S to denote two different objects; one to denote a subset of $\{1,\cdots,l\}$, of finite type and the other to denote the operator $d\partial + \partial d$ acting on $C(\mathbf{q}, \mathbf{r})$ defined in §3.4. But it is unlikely to cause confusion.

Results in (1), (2) and (3) are true for any symmetrizable Kac-Moody group G and any standard parabolic subgroup P_S of finite type. The details will appear elsewhere.

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